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RELATIVISTIC WAVE EQUATIONS IN THE DE-BROGLIE-BOHM THEORY

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ABSTRACT
The de-Broglie-Bohm theory is assumed to the a viable relativistic description of quantum mechanics and a Hamilton-Jacobi (HJ) type equation is introduced for the Schrodinger equation. An appropriate covariant generalization of the model is sought and Klein-Gordon and Dirac equations discussed. It is pointed out that, as in the classical picture, the existence of the HJ equation implies localization of the particles in space, but they could be incorporated in an extended wave phenomenon. In close analogy with the concept of general theory of relativity the particle may be regarded as a singularity and the motion of the singularity gets reaction from the quantum potential entirely different from the potential of the ordinary forces.

Key words:- Hamilton-Jacobi, Klein-Gordon, Clifford aljebra,causal-interpretation,Broglie

1. Introduction

The theory of quantum mechanics may be divided into two parts, (i) a formalism and (ii) an interpretation. The mathematical formalism of quantum mechanics has never been seriously challenged either theatrically or experimentally and remains as firmly established today as it was in the 1930's. in contrast to this, there has been controversy surrounding the interpretation of quantum mechanics since its original development[1]. Among many other, the two viable realistic descriptions of quantum mechanics are the many words interpretation and the de-Broglie-Bohm theory[2]. The object of the present talk is to provide a causal interpretation of the Klein-Gordon and Dirac equations in terms of the de-Broglie-Bohm theory and to discuss the precise way in which the quantum mechanical wave influences the evolution of the particle orbits. In the spirit of the seminar let us begin with a quote from "The character of physical law" by Richard Feynmann and end up with that if one is working from the point of view of getting beauty in one's equations, and if one has a really sound insight, one is on a sure line of progress". Feynmann's remark on QM is the following.

"There was a time when the newspapers said that only twelve men understood the theory of relativity. I do not believe there ever was such a time, on the other hand, I think it is safe to say no one understands quantum mechanics."

The theory of relativity brought about important modifications in the specific forms in which the causal laws are expressed in physics but it did not go outside the previously existing theoretical scheme. On the other hand, the quantum theory had, from the point of view of a discussion of causality, an effect that was much more revolutionary than that of relativity. In the language of Bohr, 'a radical revision of our attitude towards physical reality has been brought about by the quantum ideas'. Let us try to have a close look at the principal elements leading to such revision.

i) The entities of matter and radiation usually exist in states which do not range over a continuous spectrum of possibilities. This quantization aspect in a long and varied list of experiments.
ii) The matter and radiation are subject to a strange kind of wave-particle duality.
iii) The quantum and the wave-particle concepts force a third fundamental conclusion, the so-called uncertainty principle.

According to the uncertainty principle, an attempt to make a simultaneous measurement of where a particle is and where it is going can never be entirely successful. Thus we can no longer imagine an electron or any other small particle following a definite trajectory because no conceivable measurement can supply the simultaneous position and velocity information required to define that trajectory.

These are somewhat staggering concepts. They seem to raise more questions than they can possibly answer. How can anything be a particle and a wave at the same time? If waves are involved what kinds of waves are they? Waves of what? How can the idea of a particle motion make any sense if the particle has no trajectory? Why discard the trajectory concept? In the first palace just because it can not be measured? Whit these we tread on a very difficult ground, where the inadequacies of the conventional physical vocabulary become profoundly limiting and one takes recourse to a statistical method from the very beginning.

Many attempts have been made to construct an interpretation or a general conceptual basis for the equations of quantum physics. The efforts of Neils Bohr in this direction are particularly important. The equations of Einstein ad de-Broglie, $E = hv$ and $p = h/\lambda$ seriously disturbed Bohr since no observer ever finds a physical entity appearing as a particle and a wave at the same time. The contradictory wave and particle aspects of an electron actually appear at different times and in different kinds of experiments. Electrons as waves and as particles never appear in the same experiment. From the view point of physical observations, the only view point allowable in Physics, there is no contradiction. Waves and particles are mutually exclusive as they must be. On the other hand, wave and particle properties are both essential to a complete physical description. They are in Bohr's terminology, 'complementary' aspects of a single entity such as an electron. The so called complementarity principle is a special logical system which sets out to rationalize the entire body of the quantum phenomena. For most physicists Bohr's interpretation is acceptable. But there exists an impressive list of physicists, some of them attacking the duality doctrine and others attempting to climinate the statistical methods. Among the dissenters de Broglie and Bohm have sought an alternative interpretation of the quantum theory. This alternative interpretation helps us visualize each individual system as being in a precisely definable state, whose time evaluation is governed by a definite laws analogous to classical equations of motions (but not exactly equal to). They find that as with classical statistical mechanics, quantum mechanical probabilities may be regarded as only a practical necessity and not as a manifestation of the inherent lack of complete determinism at the quantumlevel.

2. Schrodinger Equation: Causal Interpretation

Let us first discuss the one-particle Schrodinger equation and later generalize it to the case of Klein-Gordon and Dirac equations. The one particle Schrödinger equation is

$$i\hbar \frac{\partial \Psi(\vec{x},t)}{\partial t} = -\left(\frac{\hbar^2}{2m}\right)\nabla^2\Psi(\vec{x},t) + V(x)\Psi(\vec{x},t) \quad \text{....……… (1)}$$

Madelung first recognized that Eq. (1) is equivalent to a coupled set of partial differential equations satisfied by two real functions $R(\vec{x},t)$ by

$$\Psi(\vec{x},t) = R(\vec{x},t)exp\left(\frac{i}{\hbar}S(\vec{x},t)\right). \quad \text{………. (2)}$$

Let us introduce

$$P(\vec{x},t) = [R(\vec{x},t)]^2. \quad \text{………. (3)}$$

To derive the physical significance of $S(\vec{x},t)$ and $P(\vec{x},t)$ let us write from Eqs. (1), (2) and (3).

$$\frac{\partial P}{\partial t} + \vec{\nabla}.\left(P\frac{\vec{\nabla}}{m}\right) = 0 \quad \text{… … … … … (4)}$$

and $\frac{\partial S}{\partial t}+\frac{(\vec{\nabla}S)^2}{2m}+V(\vec{x})-\frac{\hbar^2}{2m}\left[\frac{\nabla^2 P}{P}-\frac{1}{2}\frac{(\vec{\nabla}P)^2}{P^2}\right]=0.$ ………. (5)

In the classical limit $\hbar\rightarrow 0$, the above equations are subject to very simple physical interpretation. The function $S(\vec{x},t)$ is a solution of the Hamilton-Jacobi equation and $\frac{\nabla S(\vec{x},t)}{m}$ is the velocity $\vec{V}(\vec{x})$, for any particle passing the point $\vec{x}$. Thus Eq. (4) can be written as

$$\frac{\partial P}{\partial t}+\vec{\nabla}.\left(P\vec{\nabla}\right)=0\ \ldots\ldots\ldots.(6)$$

This indicates that it is consistent to regard $P(\vec{x}.t)$as a probability density for practices in our ensemble and $P\vec{\nabla}(\vec{x})$ as the corresponding mean current and Eq. (6) merely expresses the conservation of probability.

The usual statement in the causal interpretation is based on the modified Hamilton-Jacobi (HJ) equation (5) characterized by the purely quantum mechanical potential

$$U(\vec{x},t)=-\frac{\hbar^2}{2m}\left[\frac{\nabla^2 P}{P}-\frac{1}{2}\frac{\left(\vec{\nabla}P\right)^2}{P^2}\right]=-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\ \ldots\ldots\ldots\ldots(7)$$

The probabilistic element is thought to be a consequence of the lack of knowledge of the particle position and manifests itself in the presence of position probability density R^2. The main novelty of the de Broglie-Bohm theory just described consists in the emergence in the dynamical equations of the quantum potential term, which guides non-classically the motion of the particle.

Besides the presence of the additional quantum potential, the (HJ) equation in (5) has some striking deviations from its classical counterpart.

(i) In classical mechanics the Hamilton's principal function W represents a generating function that produces canonical transformation to new variables *(P, Q)* such that all new momenta *P* are constants of the motion. A complete integral of the (HJ) equation is a function *W (q, p, t)* while in the quantum case the phase of the wave function which is supposed to play the role of W is $S(\vec{x},t)$ does not depend on any constant momenta. The classical trajectory obtained from

$$mq_r=p_r=\frac{\partial W(q,\vec{p},t)}{\partial q_r}\ldots\ldots\ldots\ldots(8)$$

Depends on the specific choice of the constant momenta P and initial position $q_r(t=0)$. In contrast, the quantum trajectory s calculated, in the causal interpretation, from

$$m\vec{v}=m\vec{x}=\vec{\nabla}S\ \ldots\ldots\ldots\ (9)$$

It's solution depends on the initial position $\vec{x}(t=0)$.

(ii) The quantum potential can not be simply considered as any additional independent potential term in the (HJ) Eq. (5) because *R (x, t)* is always coupled to *S(x, t)* since only $\hbar=R\,e^{(i/\hbar)S}$ satisfies the Schrödinger equation. In fact, this coupling of R and S fieldsat the origin is of the specific form of *S(x, t)* appearing in the quantum (HJ) equation because it is this specific form of *S* that satisfies simultaneously the continuity equation. De-Broglie called equation (9) as a guidance formula which represents the main point of departure from the corresponding geometrical optics approximation or the so-called Maupertuius principle.

3. Relativistic Wave Equations

With the above background we are now in a position to extrapolate the idea of causal interpretation to relativistic wave equations[3]. Schrödinger was well aware of the limitations of the classical. $E=\frac{p^2}{2m}+V$, energy equation on which his equation is based. In fact, in his first attempts at the developmet of the wave equation, he recognized the principle of special relativity and started from the energy equation

$(E-V)^2 = C^2p^2 + m_0^2C^4$ ………. (10)
Where m_0 is the electron rest mass. Transformation of Eq. (10) to a differential form with the operators $\hat{E} = i\hbar\frac{\partial}{\partial t}$ and $\hat{p}^2 = \hbar^2\nabla^2$, leads to
$\left(i\hbar\frac{\partial}{\partial t} - V\right)^2 \Psi = \hbar^2c^2\nabla^2\Psi + m_0^2c^4\Psi$ ……….. (11)]
When equation (11) was applied to the case of hydrogen atom it produced inconsistent data and failed to reproduce experimental results. Schrödinger could find to interpretative scheme which made physical sense in all aspects of the equation. He lost faith in the equation's usefulness and even abandoned the entire wave-mechanics project for several months. Finally, he came upon the non-relativistic wave equation which we have discussed so far. The relativistic wave equation (11) with a single component represents the quantum dynamical equation for spin zero particles. Naturally, this equation known as the Klein-Gordon equation developed peculiarities in predicting data for electronic atoms. A relativistic wave equation for spin half particle was invented in 1928 by Paul Dirac. Dirac's ingeneous handling of the mathematical problem and his extra-ordanary faith in theoretical equations, even though they led hm to some practically unbelievable conclusions, make his theory an achievement that ranks with Schrödinger solving Eq. (10) Dirac wrote
$E - V = \pm C\sqrt{p^2 + m_0^2c^2}$. …………. (12)
Classical physics can make no sense of negative kinetic energy, but quantum physics admits negative values of the kinetic energy, consequently there is no reason to reject the negative sign in Eq. (12). We must be willing to accept two kinds of energies: they are conventionally called negative and positive energy states, referring to the negative and positive kinetic parts of the energy. From Eq. (12) the relativistic Hamiltonian
$H = V \pm C\sqrt{p^2 + m_0^2c^2}$ ……….. (13)
This Hamiltonain must express the total energy in operator language. No useful operator can be fashioned directly from Eq. (13) with the familiar momentum operators such as $\hat{p}_x = -i\hbar\frac{\partial}{\partial t}$, because the resulting Hamiltonian used in an energy eigenvalue equation such as $\hat{H}\Psi - E\Psi$ is decidedly nonlinear in the valuable Ψ. Some device must be invented to make equation (13) linear. Dirac did his job for us with his creative mathematical talent of an exceptional genius. In Dirac's theory Ψ wave of the electron is considered as a four component quantity $\Psi_k (k = 1,2,3,4)$ and in order to be able to form linear combinations of Ψ_k, four matrices of four rows and four columns each are introduced. As weknow the Dirac equation is given

$$\sum_{t=l}^{4} \gamma_t \left(-i\hbar\frac{\partial}{\partial x_t} + \frac{e}{c}A_t\right)\Psi_k = -im_0C\,\Psi_k\;(k = 1,2,3,4)\;\dots\dots\dots\dots(14)$$

m_0 being the rest mass of the particle. The x_t are the co-ordinates of the position four vector:
$x_1 = x, x_2 = y, x_3 = z, x_4 = i\,ct.$
The A_1 are the components of the potential four vector:
$A_1 = A_x, A_2 = A_y, A_3 = A_z, A_4 = tV.$
Using Einstein's summation convention we write Eq. (14) in the form

$$\gamma_i\left(-i\hbar\frac{\partial}{\partial x_i} + \frac{e}{c}A_i\right)\Psi = m_0C\,\Psi\dots\dots\dots\dots(15)$$

The (4×4) matrices γ_i satisfy the relation

$$\gamma_i\gamma_j + \gamma_j\,\gamma_i = 2\delta_{i\,j}\dots\dots\dots\dots.(16)$$

4. Klein-Gordon Equation: Causal Interpretation

Let us now proceed for the causal interpretation of Eqs. (11) and (16) which are the Klein-Gordon (KG) and Dirac equations respectively.
Consider the free particle Klein-Gordon equation,

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{c^2\partial\, t^2} - \frac{m_0^2c^2}{\hbar^2}\Psi = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

For our interpretation purpose this is no loss of generalization because the 'quantum mechanical' potential which causes a fundamental difference between Ψ-field and other fields is generated from the functions in Eq. (2) by only the kinetic energy term of the Hamiltonian. From equations (2) and (17) together with $P(\vec{x},t) = [R(\vec{x},t)]^2$
We get

$$\vec{\nabla}.\left(P\,\vec{\nabla}\,S\right) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(P\frac{\partial S}{\partial t}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots(18)$$

And

$$\left(\vec{\nabla}\,S\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 + m_0^2c^2 - \hbar^2\left[\frac{\nabla^2 R}{R} - \frac{1}{C^2}\frac{1}{R}\frac{\partial^2 R}{\partial\, t^2}\right] = 0 \;\ldots\ldots\ldots.(19)$$

Note that the relativistic 'quantum mechanical' potential has a form

$$U(\vec{x},t) = -\hbar^2\left[\frac{\nabla^2 R}{R} - \frac{1}{C^2}\frac{1}{R}\frac{\partial^2 R}{\partial t^2}\right] \;\ldots\ldots\ldots\ldots\ldots..(20)$$

As expected the above results represent the relativistic generalization of the continuity equation and modified Hamilton-Jacobi equation given in Eqs. (4) and (5). The transition ot the limiting case of the non-relativistic classical mechanics is quite straight-forward. In the low velocity limit the relativistic Hamiltonian

$$H = C(p^2 + m_0^2c^2)^{\frac{1}{2}}\, goes\ over\ to$$

$$H \sim m_0c^2 + \frac{p^2}{2m_0.} \;\ldots\ldots\ldots\ldots\ldots.(21)$$

Except for the rest energy m_0c^2, the result in Eq. (21) agrees with the classical non-relativistic expression for the Hamiltonian. In-as-much as the action S is related to the energy $E = -\frac{\partial s}{\partial t}$, we can go from Eqs. (18) and (19) to the corresponding non-relativistic results by introducing a new action $S^{'}$ according to the relation.

$$S = S' - m_0c^2t.\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(22)$$

From Eqs. (18) and (22) we get

$$\frac{\partial P}{\partial t} + \vec{\nabla}\left(P\frac{\vec{\nabla}S'}{m_0}\right) - \frac{1}{m_0c^2}\left(\frac{\partial P}{\partial t}\frac{\partial S'}{\partial t} + P\frac{\partial^2 S'}{\partial t^2}\right) = 0 \quad \ldots\ldots\ldots\ldots\ldots.(23)$$

And

$$\frac{\partial S'}{\partial t} + \frac{\left(\vec{\nabla}S'\right)^2}{2m_0} - \frac{1}{2m_0c^2}\left(\frac{\partial S'}{\partial t}\right)^2 - \frac{\hbar^2}{2m_0}\left[\frac{\nabla^2 R}{R} - \frac{1}{c^2}\frac{1}{R}\frac{\partial^2 R}{\partial t^2}\right] = 0.\ldots\ldots\ldots\ldots(24)$$

In the non-relativistic case $c \to \infty$ Eqs. (23) and (24) are in exact agreement with Eqs. (4) and (5). This analysis shows that except for relativistic modifications of the associated Hamilton-Jacobi equation and continuity equation, de Brogle-Bohm theory leads to physical interpretation of the Klein-Gordon equation, which is equation the density, and the guidance formula are

$$\rho = P\frac{\partial S}{\partial t}$$

$$\vec{v} = c^2\vec{\nabla}S/\frac{\partial S}{\partial t} \ldots\ldots\ldots.(25)$$

If instead of a free particle, we considered a particle subjected to an electromagnetic field derivable from the potentials V and $\vec{A}$ we would arrive at

$$\rho = P\left(\frac{\partial S}{\partial t} - eV\right) \; and \; \vec{v} = -c^2 \frac{\vec{\nabla} S + \frac{e}{c}\vec{A}}{\frac{\partial S}{\partial t} - eV} \ldots\ldots\ldots\ldots\ldots (26)$$

In this case the real function S would satisfy

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t} - e\,V\right)^2 - \sum_{x,y,z}\left(\frac{\partial s}{\partial x} + \frac{e}{c}A_x\right)^2 = m_0^2c^2 + \hbar^2\frac{\Box R}{R}.\ldots\ldots\ldots\ldots (27)$$

Let us rewrite (27) as

$$\frac{1}{c^2}\left(\frac{\partial s}{\partial t} - e\,V\right)^2 - \sum_{x,y,z}\left(\frac{\partial s}{\partial x} + \frac{e}{c}A_x\right)^2 = M_0^2c^2.\ldots\ldots\ldots (28)$$

Where M_0 is a variable rest mass given by

$$M_0 = \sqrt{m_0^2 + \frac{\hbar^2}{c^2}\left(\frac{\Box R}{R}\right)} \ldots\ldots\ldots\ldots\ldots (29)$$

Appearance of the quantum potential in M_0 was realized by de Broglie as follows.

"The existence of a Hamiltonian-Jocvobi equation implies that the particles are localized in space, as in the classical picture, but they would be incorporated in an extended wave phenomenon. The particle is subject to the action of forces exerted on it in the course of its trajectory. With general theory of Relativity de-Broglie regarded the particle as a singularity which serves as a centre of extended wave phcnomonon. The motion of the singularity gets reaction from the quantum potential entirely different from the povential of the ordinary forces. "

It is to transcrive the (KG) equation's result in the formalism of general theory of relativity. The wave equation is given by

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial_x k}\left[\sqrt{-g}g^{kl}\frac{\partial \Psi}{\partial_x k}\right] - \frac{2i}{\hbar}P^k\frac{\partial \Psi}{\partial_x} + \frac{1}{\hbar^2}\left(m_0^2c^2 - \frac{e^2}{c^2}p^2\right)\Psi = 0 \ldots\ldots\ldots\ldots\ldots (30)$$

Where g_{ik} 's are t he classical coefficients of the metric of space time. g^{ik} 's are the corresponding contravarient component and $g = det(g_{uk})$. P^k's are the components of four vector potential, $P^2 = P_kP^k = V^2 - A^2$. The operator $\frac{1}{\sqrt{-g}}\frac{\partial}{\partial_k}\left[\sqrt{-g}g^{kl}\frac{\partial}{\partial_{xl}}\right]$ is the generalization of the D' Alembertian. In the tensor notation inftroduced above the modified Hamilton-D'Alemberitain. In the tensor notation introduced above the mdifed Hamiloton. Jocobi equatt9h (28) reads

$$g^{k\,l}\left(\frac{\partial s}{\partial_x k} - e\,p_k\right)\left(\frac{\partial s}{\partial_x l} - e\,P_l\right) = m_0^2c^2 \quad \ldots\ldots\ldots\ldots..(31)$$

In view of Eq. (31) the four vector velocity will be given by

$$M_0\,cu' = g^{k\,l}\left(\frac{\partial s}{\partial_x k} - eP_x\right)\ldots\ldots\ldots\ldots\ldots..(32)$$

Eq. (32) represents the general form of guidance formula.

5. Dirac Equation: Casual Interpretation

For a possible causal interpretation of the Dirac equation we shall first introduce the so called Gordon decomposition of the current density four vector j_t of Dirac's theory. Let us introduce $\Psi_k^+ = \Psi_k^*\,\gamma_4$ or symbolically$\Psi^+ = \Psi^*\gamma_4$. The wave equation for Ψ^+ is

$$\left(i\hbar\frac{\partial}{\partial x_i} + \frac{e}{c}\,A_i\right)\Psi^+\gamma_4 = -im_0c\,\Psi^+.\ldots\ldots\ldots\ldots..(33)$$

From Eqs. (16) and (33) we have

$$\frac{\partial}{\partial x_i}(\Psi^{+}\gamma_i\Psi) = 0 \ldots\ldots\ldots\ldots..(34)$$

Thus we write the current density four vector as

$$j_i = i\, c\, \Psi^{+}\, \gamma_i = i\, c\, \sum_{\substack{k=1\\ l=1}}^{4} \Psi_k^{*}\,(\gamma_4\gamma_i)_{k\,l}\Psi_l. \ldots\ldots\ldots\ldots\ldots\ldots..(35)$$

The first three components of j_i are

$$\vec{j} = i\, c\, \Psi^{*}\overrightarrow{\gamma\Psi} = -c\Psi^{+}\overrightarrow{\alpha\Psi} \ldots\ldots\ldots\ldots.\ \ (36)$$

Eq. (36) gives us the components of the particle current density along space axis while the fourth component

$$\rho = \sum_{k=l}^{4} \Psi_k^{*}\Psi_k = \sum_{k=l}^{4} \left|\Psi_k\right|^{2} \ldots\ldots\ldots.(38)$$

From eqs. (15), (16) and (33) we can derive

$$j_l = \frac{i\hbar}{2\, m_v}\left(\Psi^{+}\frac{\partial\Psi}{\partial x_l} - \frac{\partial\Psi^{+}}{\partial x_l}\Psi\right) = -\frac{e}{m_{0\,c}}\, A_l\, \Psi^{+}\Psi - \frac{i\hbar}{2m_o}\sum_{i\neq t}\frac{\partial}{\partial x_i}(\gamma^{+}\gamma_i\,\gamma_t\,\Psi) \ldots..\ \ (39)$$

We can thus break up the four vector j_t into two four vectors $j_t^{(1)}$ and $j_t^{(2)}$ defined by

$$j_t^{(1)} = \frac{i\hbar}{2m_0}\left(\Psi^{+}\frac{\partial\Psi}{\partial x_t} - \frac{\partial\Psi^{+}}{\partial x_l}\Psi\right) - \frac{e}{m_0 c}A_l\Psi^{+}\Psi \ldots\ldots\ldots..(40)$$

$$j_t^{(2)} = -\frac{i\hbar}{2m_0}\sum_{i\neq l}\frac{\partial}{\partial x_i}(\Psi^{+}\gamma_i\,\gamma_l\,\Psi) \ldots\ldots\ldots\ldots(41)$$

The decomposition of j_t into $j_t^{(1)}$ and $j_t^{(2)}$ goes by the name Gordon's decomposition. Physically $j_t^{(1)}$ arises from the overall motion (orbital) of the particle and $j_t^{(2)}$ from the particle spin.

In the cases of Schrödinger and KG equations where we had a one component wave function it was sufficient to consider $\Psi(\vec{x},t) = R(\vec{x},t)\, exp\frac{i}{\hbar}S(\vec{x},t)$ but for the Dirac equation we run into difficulties. For the four component Ψ in Dirac's theory we have to write

$$\Psi_k(x,y,z,t) = R_k(x,y,z,t)exp\left(\frac{i}{\hbar}S_k(x,y,z,t)\right) \ldots\ldots\ldots\ldots(42)$$

It is not all clear that all S_k's should be the same. However, Vigier[4] has chosen to work with a common phase *S(x, y, z, t)* and arrived at a guidance formula by dealing with the overall current density four vector j_l.

Substituting Eq. (42) in Eq. (39) we get

$$j_l = \frac{1}{m_0}\sum_{k=1}^{4}\left(\frac{\partial S_k}{\partial x_l} + \frac{e}{c}A_l\right)R_k^{+}R_k - \frac{i\hbar}{2m_0}\sum_{i\neq l}\frac{\partial}{\partial x_l}(R_k^{+}\gamma_i\gamma_l R_k) \ldots\ldots\ldots.(43)$$

With $R_k^{+} = R_k\gamma_4$

The motion of a particle may be defined by its four velocity. To make the guidance of the particle by the Ψ wave perfectly clear seems natural to make the Ψ wave follow one of the lines of the current defined by the j_l vector, that is to put

$$u_i = Kj_{i^{+}} \ldots\ldots\ldots\ldots\ldots(44)$$

But the four vector u_i obeys the relation $\sum_{i=l}^{4} u_i^2 = -1$, from which we conclude

$$K = \frac{2}{\sqrt{-\sum_{i=l}^{4} j_l^2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(45)$$

Vigier proposed to define the variable proper mass M_0 of the particle by the formula

$$M_0 c\, u_i = \frac{m_0 j_i}{\Psi^+\Psi} = \frac{m_0 j_i}{R^+R} \ldots\ldots\ldots\ldots\ldots (46)$$

Thus

$$M_0 = \frac{m_0}{R^+R}\frac{j_i}{cu_i} = \frac{m_0}{R^+R}\frac{1}{c}\sqrt{-\sum_{k=1}^{4} j_k^2} \ldots\ldots\ldots\ldots\ldots (47)$$

The 'M_0' in Eq. (47) is a generalization within the Dirac theory of the definition that we adopted in relativistic wave mechanics for a single component. To be more specific let us introduce

$$\left(\frac{\partial \bar{S}}{\partial x_j}\right) = \frac{\sum_{k=1}^{4} \frac{\partial S}{\partial x_j} R_k^+ R_k}{\sum_{k=1}^{4} R_k^+ R_k} \ldots\ldots\ldots\ldots (48)$$

and

$$P_j = -\frac{e}{c}A_j + \frac{\frac{\hbar}{i}\sum_{i\neq j}\sum_{k=1}^{4}\frac{\partial}{\partial x_i}\left(R_k^+\gamma_i\gamma_j R_k\right)}{\sum_{k=1}^{4} R_k^+ R_k}$$

$$= -\frac{e}{c}A_j + \frac{j_j^{(2)}}{R^+R} m_0 \ldots\ldots\ldots (49)$$

From Eqs.(39), (44) and (49) we get

$$M_0\ cu_i = -\frac{\partial \bar{S}}{\partial x_i} + P_i = -\frac{\partial \bar{S}}{\partial x_i} - \frac{e}{c}A_i + P_i' \ldots\ldots\ldots\ldots\ldots (50)$$

Equation (50) plays the role of the guidance formula in the Dirac's theory. The influence of the spin on the motion is represented by the term P_i' proportional to $j_l{}^2$.

In order to move from the Dirac to the (KG) theory we must first neglect the effect of spin i.e. the term P_i' and assume that the Ψ_k reduce to single component Ψ.

Adapting the tensor notation one then gets back the formula

$$M_0\ c\ u^l = g^{kl}\left(\frac{\partial S}{\partial x^k} - eP_k\right) \ldots\ldots\ldots\ldots (51)$$

Further we note that for a spinless particle with a single component of , $j_l^{(1)}$ only survives and we have

$$\sum_{i=1}^{4} j_l^{(1)^2} = \frac{R^4}{m_0^2}\sum_{i=1}^{4}\left(\frac{\partial S}{\partial x_i} - \frac{e}{c}A_i\right)^2 \ldots\ldots\ldots\ldots\ldots (52)$$

Substituting Eq. (52) in Eq. (47) we find

$$M_0^2\ c^2 = -\sum_{i=l}^{4}\left(\frac{\partial S}{\partial x_i} - \frac{e}{c}A_i\right)^2 \ldots\ldots\ldots\ldots (53)$$

This is the relativistic Hamilton-Jacobi equation for the Klein-Gordon equation.

6. We have considered here only body equation. Recently it has been shown how the causal interpretation of a spin half particle described by a Pauli Spinor may be extended to treat the two-body case[5]. All the degrees of freedom in the wave function have been interpreted in terms of interconnected Eucledian tensors generated by a direct product of Clifford algebras.

To my knowledge the causal interpretation has not yet been extended to deal with the spin 1 and spin 3/2 particles. Admittedly, such a problem will be mathematically quite complicated but nevertheless interesting.

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