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COMPREHENSIVE STABILITY ANALYSIS OF ROTATING TAPERED BEAMS UNDER COMBINED THERMAL GRADIENTS AND AXIAL COMPRESSION

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Abstract. The static stability of a rotating, non-uniform tapered beam subjected to axial excitation under a thermal gradient is investigated for both clamped–clamped and pinned–pinned boundary conditions. The equations of motion and corresponding boundary conditions are derived using the extended Hamilton's principle. By applying the extended Galerkin method to the non-dimensionalized equations, a system of Hill's equations is formulated. Static buckling loads are then determined from these equations. The study examines how various parameters including geometric configuration, thermal gradient, taper ratio, and rotational speed affect the buckling behavior of the beam. Numerical solutions and graphical results are obtained using a MATLAB-based computational code. The findings indicate that increased rotational speed and taper enhance stability, while a higher thermal gradient tends to reduce it, regardless of the boundary condition.

Keywords. Static Stability, Galerkin method, Hamilton's principle, taper parameter, temperature parameter.

1. Introduction

The stability analysis of rotating cantilever beams with axial orientation perpendicular to the spin axis holds significant practical relevance in mechanical engineering. Numerous engineering systems can be modeled as rotating cantilever beams, including turbomachinery and turbine blades, helicopter rotor blades, end mills and boring bars in machining operations, aircraft propellers, flexible spacecraft appendages, satellite antennas, and robotic manipulators. There has been many research on the stability analysis of uniform rotating beam [1-6]. On the other hand, the study on the stability of rotating beams with tapered cross-section have been carried out comparatively in the recent past. [7-15].

Vibrations of a rotating cantilever beam having uniform cross section with a tip mass was investigated by Bhat[1] using the Rayleigh-Ritz routine based on beam characteristic orthogonal polynomials. Unger and Brull[2] studied the parametric instability of a rotating shaft subjected to pulsating torque applied at the ends. The effect of spin speed and the hub radius of a rotating Euler beam on its vibration and buckling was studied by Bauer and Eidel[3]. Yang et al.[4] studied the dynamic modelling of a rotating Euler-Bernoulli beam using the extended Hamilton's principle. Sinha[5] analyzed the transient vibration of a rotating beam subjected to an intermittent pulse load at the tip of the free end based on Timoshenko theory. Huang et.al.[6] studied the free vibration of rotating cantilever Euler beams at high angular velocity with solution based on power series method.

Study of the literature reveals that very few work has been done on the stability analysis of rotating tapered beams. The vibration of a non-uniform rotating beam with a restrained base for different values of rotational speed was investigated by Liu and Yeh[7]. Kim et al.[8][9] developed and investigated the static failure of a tapered, filament wound rotating Timoshenko shaft about its axis under free vibration and under forced vibration respectively. Banerjee et al.[10] studied the free vibration of rotating tapered Euler-Bernoulli beam having two cases of cross-sections viz., first: constant width and linearly varying depth, second: linearly varying width and depth. Akgöz and Civalek[11] investigated the free vibration of non-homogeneous tapered cantilever Euler Bernoulli micro beams for various taper ratios based modified coupled stress theory. The static stability and vibration of circularly tapered functionally graded material beam subjected to constant axial compressive force based on Euler Bernoulli beam theory have been studied by Mazzei Jr and Scott[12]. They considered linear, sinusoidal and exponential type of tapered beams. Shahba and Rajasekaran[13] studied the stability and vibration of tapered axially functionally graded Euler Bernoulli beams. The free vibration study of a rotating tapered Rayleigh beam was carried out by Banerjee and Jackson[14]. The divergence instability mechanism of a rotating Timoshenko beam with precone was studied by Lee et al.[15].

Khammer and Schlack[16] studied the effects of a non-constant angular velocity upon the vibration of a rotating Euler beam using Krylov–Bogoliubov–Mitropolsky (KBM) perturbation method. The stability analysis of a simply supported rotating shaft having asymmetric cross-section under a harmonic axial thrust and stochastic load was studied by Namachchivaya[17]. He referred approximated Markov model to deal with the stochastic part and followed Routh-Hurwitz criterion for the first and second moment stability. The stability analysis of a rotating Timoshenko beam with a flexible root under periodic axial force was investigated by Abbas[18] using finite element method. Ishida et al.[19] discussed the vibration and stability of a parametrically excited rotating shaft under sinusoidal axial force. Chen[20] investigated the parametric instability of a twisted Timoshenko beam acted upon by an axial pulsating force. Dash et.al.[21] studied the static instability of an asymmetric rotating sandwich beam under an axial pulsating load. Hybrid basis functions were derived for the finite element vibration analysis of high speed rotating tapered beams by Gunda et al.[22]. Banerjee and Kennedy[23] investigated the in-plane free vibration of rotating uniform Euler-Bernoulli beam using dynamic stiffness method and the influence of Coriolis effects, hub radius, rotational speed was studied.

Very few research has been done on the stability of rotating beam under thermal gradient. Nayak et.al.[24] studied the static stability analysis of an asymmetric sandwich beam with thermal gradient subjected to an axial pulsating load. Librescu et.al.[25] investigated the stability of rotating turbo machinery blades operating in high temperature environment. However, no research is reported in the literature on the stability analysis of rotating tapered beam under thermal gradient. In current paper the governing differential equations of motion of a circularly tapered Euler-Bernoulli beam subjected to axial pulsating load has been derived with an effect of one dimensional temperature gradient along the central length of the beam.

2. Formulation of the problem

A rotating tapered cantilever beam of length l set off a distance C_0 from the axis of rotation and rotating at a uniform angular velocity Ω about a vertical z'-axis and is capable of oscillating in the x-z plane. The beam is oriented along the x-axis perpendicular to the axis of rotation as shown in Fig.1 below. A pulsating axial force $P(t) = P_0 + P_1 \cos \omega t$ is applied at the end $x = C_0 + l$ of the beam along the point of C.G. of the cross-section in the axial direction, ω being the frequency of the applied load, t being the time and P_0 and P_1 being respectively the static and dynamic load amplitudes. The assumptions made for establishing the differential equations of motion are as follows:

(a) The material of the beam is homogeneous & isotropic in nature.
(b) The deflections of the beam are small and the transverse deflection $w(x,t)$ is same for every points of a cross-sectional area.

(c) The beam obeys Euler-Bernoulli beam theory.
(d) Extensional deflection of the beam is neglected.
(e) A steady one-dimensional temperature gradient is assumed along the axial direction of the beam through the central line.
(f) Extension and rotary inertia effects are negligible.

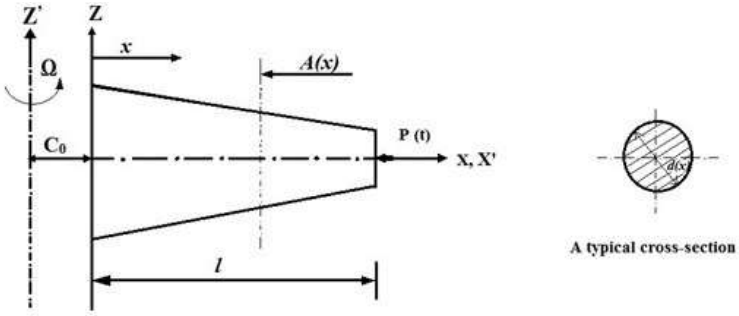


Figure. 1. *System Configuration*

The potential energy, kinetic energy and work done of the rotating beam are derived as under:

$$V=\frac{1}{2}\int_0^l E(x)I(x)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx+\frac{1}{2}\int_0^l \rho A(x)\Omega^2(C_0+x)\int_0^{x'}\left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (1)$$

$$T=\frac{1}{2}\int_0^l \rho A(x)\left(\frac{\partial w}{\partial t}\right)^2 dx+\frac{1}{2}\int_0^l \rho\Omega^2 A(x)w^2 dx \qquad (2)$$

$$W_P=\frac{1}{2}\int_0^l P(t)\left(\frac{\partial w}{\partial x}\right)^2 dx \qquad (3)$$

where, $w(x,t)$ is transverse deflection of the beam.

The application of the extended Hamilton's principle gives the following equation of motion and boundary conditions

$$\delta\int_{t_1}^{t_2}(T-V+W_P)=0 \qquad (4)$$

$$\left[E(x)I(x)w,_{xx}\right],_{xx}+\rho A(x)w,_{tt}+\rho\Omega^2 I(x).w,_{xx}-\left[N(x_1)w,_x\right],_x+P(t)w,_{xx}=0 \quad (5)$$

where,

$$N(x_1)=\frac{1}{2}\rho A(x)\Omega^2\left[(C_0+l)^2-(C_0+x')^2\right]$$

The boundary conditions at x = C_0 and x = (C_0+ l) are

$$\left[E(x)I(x)w,_{xx}\right],_x+P(t)w,_x=0,$$

$$\left[E(x)I(x)w,_{xx}\right]_{x=l}=0,$$

$$w,_t=0. \qquad (6)$$

where, $w_{,x} = \frac{\partial w}{\partial x}, w_{,xx} = \frac{\partial^2 w}{\partial x^2}, w_{,t} = \frac{\partial w}{\partial t}, w_{,tt} = \frac{\partial^2 w}{\partial t^2}$

Introducing dimensionless parameters

$$\xi = \frac{x}{l}, \eta = \frac{w}{l}, c_0 = \frac{C_0}{l}, \tau = ct \because c^2 = \frac{E(x)I(x)}{\rho A(x) l^4}$$

The non-dimensional equation of motion and boundary conditions can be written as,

$$\left[S(\xi)T(\xi)\eta''\right]'' + m(\xi)\ddot{\eta} + \left[r_g \Omega_0^{\,2} + p(\tau)\right]\eta'' - \Omega_0^{\,2}\left[q(\xi)\eta'\right]' = 0 \qquad (7)$$

and,

$$\left\{\left[S(\xi)T(\xi)\eta''\right]' + p(\tau)\eta'\right\}_{\xi=1} = 0,$$

$$\left[S(\xi)T(\xi)\eta''\right]_{\xi=1} = 0,$$

$$\eta(0,\tau) = 0,$$

$$\eta'(0,\tau) = 0. \qquad (8)$$

2.1. Approximate Solution

Approximate solution to the non-dimensional equations of motion are assumed as

$$\eta(\xi,\tau) = \sum_{r=1}^{N} \eta_r(\xi) f_r(\tau) \qquad (9)$$

where, $f_r(\tau)$ is an unknown function of time and $\eta_r(\xi)$ is a coordinate function to be so chosen as to satisfy as many of the boundary conditions in Eq. (4) as possible. It is further assumed that $\eta_r(\xi)$ can be represented by a set of functions (3) which satisfy the conditions obtained from Eq. (4) by deleting the terms containing ω_0 and $p(\tau)$. It is further assumed that

Table.1. *The coordinate functions*

End Arrangement	Coordinate Functions $i = 1, 2, ..., r$
Pinned-Pinned	$\eta(\xi) = \sin(\pi i \xi)$
Clamped-Clamped	$\eta(\xi) = \xi^{(i+1)} + 2\xi^{(i+2)} + \xi^{(i+3)}$
Guided-Pinned	$\eta(\xi) = \cos\{(2i-1)\pi\xi/2\}$
Clamped-Pinned	$\eta(\xi) = 2(i+2)\xi^{(i+1)} - (4i+6)\xi^{(i+2)} + 2(i+1)\xi^{(i+3)}$
Clamped- CUR	$\eta(\xi) = (i+3)(i+2)^2(i+1)\left\{\xi^{(i+1)} - 2\xi^{(i+2)} + \xi^{(i+3)}\right\}$
Clamped- Free	$\eta(\xi) = (i+2)(i+3)\xi^{(i+1)} - 2i(i+3)\xi^{(i+2)} + i(i+1)\xi^{(i+2)}$

coordinate functions for the various boundary conditions can be approximated by the ones given in Table 1.

Substitution of the series of solutions in the non-dimensional equations of motion and subsequent application of the general Galerkin method[24] leads to the following matrix equations of motion:

$$[M]\{\ddot{f}\}+[K]\{f\}-\{p_0[H]-p_1\cos\theta\tau[H]\}\{f\}=\{0\} \qquad (10)$$

The various matrix elements are given by

$$\int_0^1 m(\xi)\eta_i(\xi)\eta_j(\xi)\,d\xi = M_{ij}$$

$$\int_0^1 \left\{S(\xi)T(\xi)\eta_i''(\xi)\eta_j''(\xi)+\Omega_0^2\left[q(\xi)-r_g\right]\eta_i'(\xi)\eta_j'(\xi)\right\}d\xi = K_{ij}$$

$$\int_0^1 \eta_i'(\xi)\eta_j'(\xi)\,d\xi = H_{ij}$$

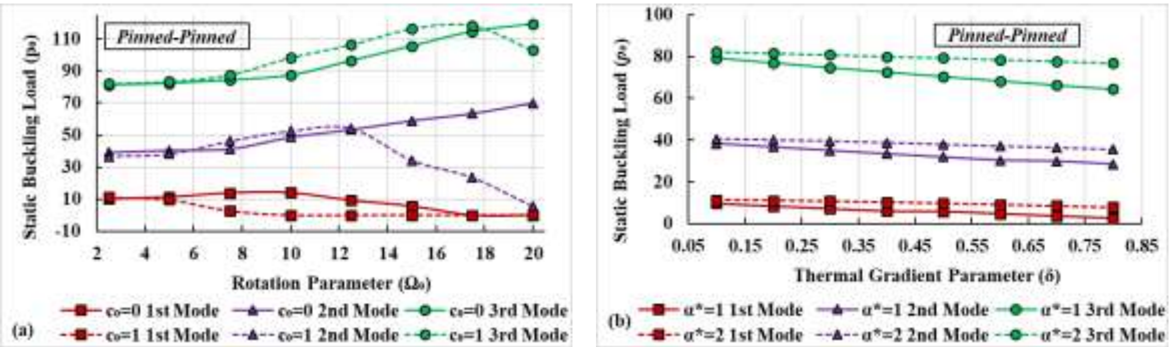
$$\because i,\ j = 1,2,\ldots\ldots\ldots,N$$

2.2. Static Buckling Loads

Substitution of p_1 =0 and $\{\ddot{f}\}=\{0\}$ in Eq. (10) leads to the eigenvalue problem $[K]^{-1}[H]\{f\}=\dfrac{1}{p_0}\{f\}$. The static buckling loads for the first few modes are obtained as the reciprocals of the eigenvalues of $[K]^{-1}[H]$ using MATLAB R2013a reference guide, version 8.1.0.604, 15 February 2013.

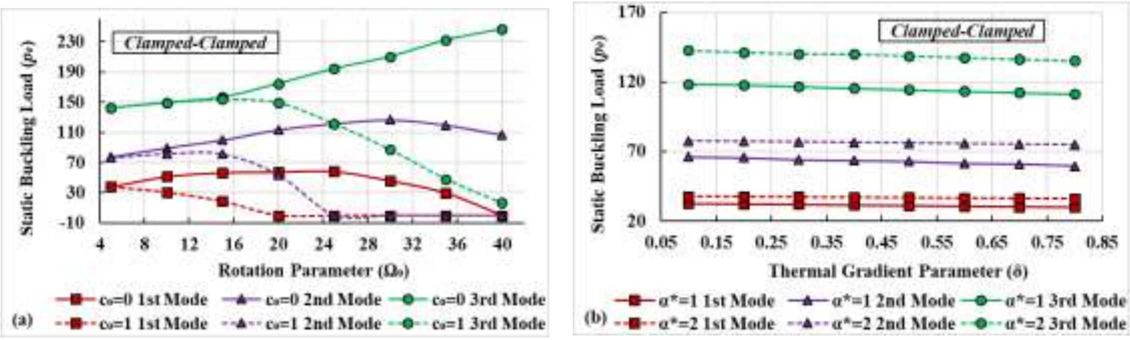
3. Numerical Results and Discussions

Numerical results were obtained for various values of the parameters like rotation parameter, geometric parameter, taper parameter and thermal gradient. The linearly tapered cantilever beam with a circular cross-section is assumed to have a diameter varying according to the relation



***(a)** For c_0=0 & 1, with δ=0.1, α^*=2* ***(b)** For α^*=1 & 2, with Ω_0=5*

***Figure.2.** Variation of static buckling loads of pinned-pinned beam at first three modes*



***(a)** For c_0=0 & 1, with δ=0.1, α^*=2* ***(b)** For α^*=1 & 2, with Ω_0=5*

***Figure.3.** Variation of static buckling loads of clamped-clamped beam at first three modes*

$$d(\xi) = d_1\left[1+\alpha^*(1-\xi)\right]$$

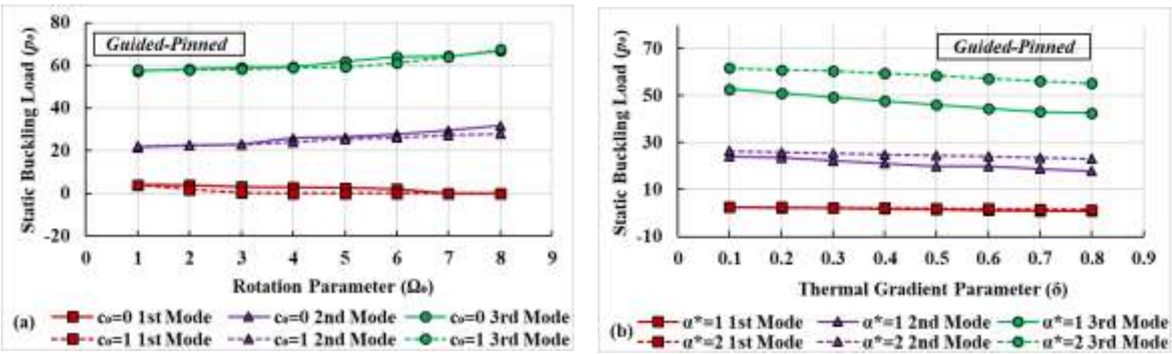
where, d_1 is the diameter of the beam at the end $\xi = 1$ and α^* is the diameter taper parameter. Consequently, the mass distribution $m(\xi)$ and the moment of inertia distribution $S(\xi)$ are given by the relations

$$m(\xi) = \left[1+\alpha^*(1-\xi)\right]^2$$

$$S(\xi) = \left[1+\alpha^*(1-\xi)\right]^4$$

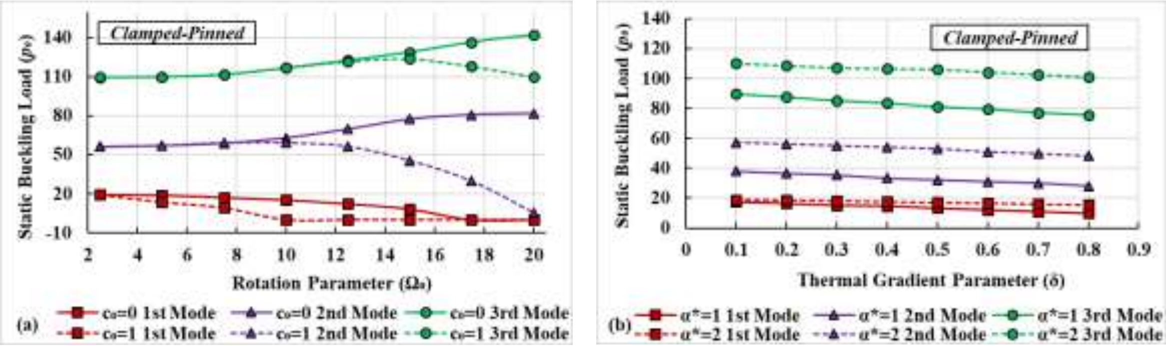
The temperature above the reference temperature at any point ξ from the end of the beam is assumed to be $\psi = \psi_0(1-\xi)$. Choosing $\psi = \psi_1$, the temperature at the end $\xi = 1$ as the reference temperature, the variation of modulus of elasticity of the beam is denoted by

$$E(\xi) = E_1\left[1-\gamma\psi_1(1-\xi)\right] = E_1 T(\xi), 0 \le \gamma\psi_1 \le 1$$



***(a)** Effect of Ω_0, with δ=0.1, α^*=2* ***(b)** Effect of δ, with Ω_0=5*

***Figure.4.** Variation of static buckling loads of guided-pinned beam at first three modes*



***(a)** Effect of Ω_0, with δ=0.1, α^*=2*

***(b)** Effect of δ, with Ω_0=5*

***Figure.5.** Variation of static buckling loads of clamped-pinned beam at first three modes*

where, γ is the coefficient of thermal expansion of the beam material, $\delta = \gamma\psi_1$ is the thermal gradient parameter and

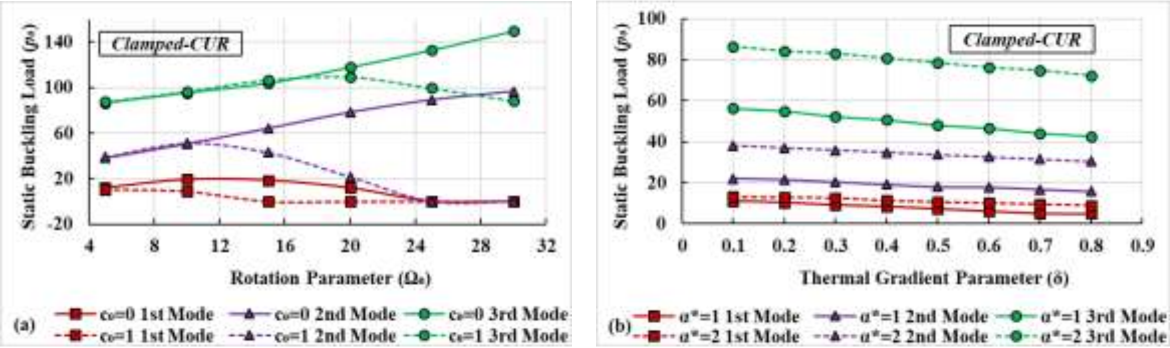
$$T(\xi) = \left[1 - \delta(1-\xi)\right]$$

where, δ is the thermal gradient along the length of the beam.

The static stability analysis of the system for various boundary conditions has been analyzed as follows:

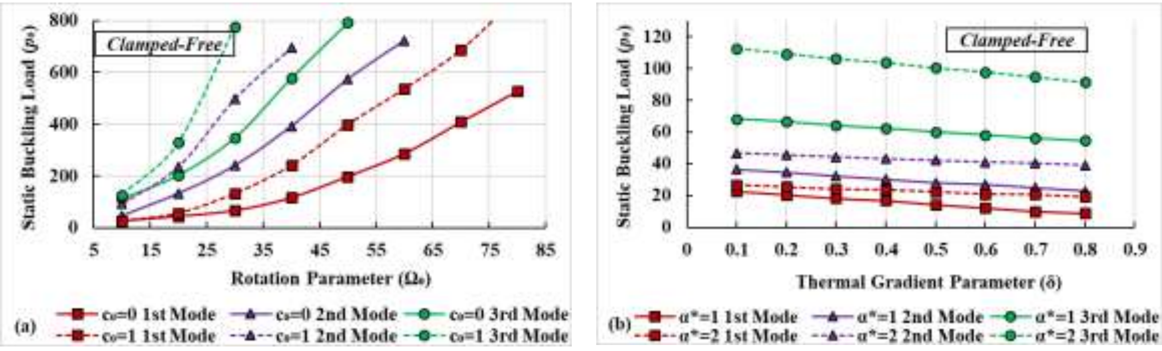
The static stability of the system has been analyzed and are presented as functions of rotational speed parameter Ω_0 for two different values of the hub radius c_0 in Figs. 2(a), 3(a), 4(a), 5(a), 6(a) and 7(a) for boundary conditions of pinned-pinned, clamped-clamped, guided-pinned, clamped-pinned, clamped-CUR and clamped-free respectively. The figures show the effect of rotational speed on the static buckling loads of the first three modes of a beam for $c_0 = 0$ and $c_0 = 1$. It is found that with $c_0 = 0$, while the buckling loads of the first mode of a rotating beam having combinations of clamped, guided and pinned ends (except guided-pinned) increase with increase in the value of Ω_0 to reach a maximum and then decrease, those of the third mode increase monotonically with Ω_0 in the range of values considered.

Further, the behavior of the buckling load characteristic of the second mode for clamped-clamped condition is analogous to that of the first mode, whereas, for pinned-pinned and clamped-pinned cases, it is similar to that of the third mode. For $c_0 = 1$, the buckling loads of all the modes decrease with increase in rotational speed beyond a certain optimum value so as to attain or tend to attain the value zero. For the aforementioned boundary conditions, the buckling load for $c_0 = 1$ may be either higher or lower than that for $c_0 = 0$, depending on the rotational speed. Moreover, the rotational speed at which the buckling load attains zero value decreases with increase in the hub radius. On the other hand, the buckling loads of all the three modes of beams having a free end at $\xi = (1 + c_0)$ increase with increase in both Ω_0 and c_0.



(a) *Effect of Ω_0, with δ=0.1, α^*=2* ***(b)*** *Effect of δ, with Ω_0=5*

Figure.6. *Variation of static buckling loads of clamped-CUR beam at first three modes*



(a) *Effect of Ω_0, with δ=0.1, α^*=2* ***(b)*** *Effect of δ, with Ω_0=5*

Figure.7. *Variation of static buckling loads of clamped-free beam at first three modes*

In Figs. 2(a), 3(a), 5(a) and 6(a) the static buckling loads have a greater value for higher modes with $c_0 = 0$ and for $c_0 = 1$ it increases up to certain range of rotational speed parameter and then decreases. Figures 4(a) and 7(a) show little difference from the other four boundary conditions. For the first case static buckling loads show equal values for both the values of c_0 for higher modes and for first mode it is having a constant value over considered range of Ω_0 for $c_0 = 0$ and very poor values for $c_0 = 1$. For the second case, static buckling loads increases monotonically for both $c_0 = 0$ and $c_0 = 1$ for all the three modes.

Figures 2(b), 3(b), 4(b), 5(b), 6(b) and 7(b) show the variations of static buckling loads of the first three modes as a function of the thermal gradient parameter δ for the two different combinations of the taper parameter α^*. It has been observed that for a given tapered beam, the value of static buckling load decrease monotonically with increase in the value of δ, the rate of decrease being greater for higher modes. Whereas, at any value of δ, the static buckling loads record slightly larger changes in higher modes with an increase in the value of the taper parameter.

In the first mode, static buckling loads have nearly equal values for both the values of taper parameter for all cases. For higher modes, the values of static buckling loads are higher for third mode than second mode and vary linearly towards lower value with the increase in thermal gradient.

4. Conclusion

In this paper a computational analysis of the static stability of a tapered cantilever beam with axial load and thermal gradient under pinned-pinned & clamped-clamped boundary conditions are considered. The programming has been developed by in MATLAB environment. The following are the conclusions drawn from the study.

The static stability of a rotating tapered beam under a pulsating axial load is investigated for possible combinations of clamped, guided, pinned and free boundary conditions. It is observed that beam with clamped-free condition becomes statically stable. Beams with the other end conditions may either stabilize or destabilize with increase in rotational speed and hub radius. Increase in taper parameter increases the static buckling loads. However, increase in thermal gradient reduces the static buckling loads. Thus, it may be inferred that increasing taper have stabilizing effects on the beams, whereas increasing temperature gradient have a destabilizing effect on the beams for both the cases.

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