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ON FOLDNESS OF NEUTROSOPHIC FUZZY H-IDEAL IN BCK-ALGEBRA

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ABSTRACT
This paper introduces neutrosophic fuzzy n-fold H-ideal in BCK-algebra, deriving foundational properties, theorems that pave the way for advancements in algebraic structures and fuzzy logic.
Keywords: Neutrosophic Fuzzy Sets (NFS's), BCK-algebras, n-fold H-ideal (nHI), Neutrosophic Fuzzy n-fold H-Ideal (NFnHI), Neutrosophic Fuzzy n-fold closed H-ideal (NFnCLHI).
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I. Introduction
Iseki & Tanaka [5, 6] introduced BCK-algebraic theory introduction and also develops ideal theory of BCK-algebras. Zadeh's fuzzy sets [12] generalized crisp sets. Fuzzy set contains degree of truth membership. [2, 3] Atanassov's introduced the novel concept of fuzzy that is an intuitionistic fuzzy sets, this type of fuzzy sets contain degree of truth membership and degree of non-memberships. In this area many researchers are given number of inventories research papers. In [8] Jun et.al., introduced an intuitionistic fuzzy ideals in BCK-algebras. In [9, 10] Satyanarayana et.al., generalized the concept of product and direct product of intuitionistic fuzzy BCK-algebras. [7] Zhan and Tan's introduced characterization of fuzzy H-ideals in BCK-algebras in doubt manner of fuzzy sets. In 2025, [1, 4] authors developed a novel concept of an interval-valued neutrosophic fuzzy hyper BCK-ideal and implicative hyper BCK-ideal of hyper BCK-algebras and also introduced derivations & translations of neutrosophic fuzzy positive implicative ideals of BCK-algebra. Subsequent studies by Satyanarayana et.al., [11] examined foldness of intuitionistic fuzzy H-ideals in BCK-algebra. We generalized the advance field with Neutrosophic Fuzzy n-fold H-ideals (NFnHI) in BCK-algebras.
In this article, we introduced neutrosophic fuzzy n-fold H-ideal in BCK-algebra, and studied their fundamental properties.
In this paper, we used the following abbreviations:
- BCK-A (or) $\mathfrak{A}$: BCK-algebras
- NFnHI : neutrosophic fuzzy n-fold H-ideals
- NFS : Neutrosophic fuzzy sets
- NFI : neutrosophic fuzzy ideals
- nHI : n-fold H-ideals
- NFnCLHI : neutrosophic fuzzy n-fold closed H-ideals
- IFS : intuitionistic fuzzy set
- IFI : intuitionistic fuzzy ideals
- IFCLI : intuitionistic fuzzy closed ideals
- FHI : fuzzy H-ideals
- IFHI : intuitionistic fuzzy H-ideals
- IFnHI : intuitionistic fuzzy n-fold H-ideals
- IFCLHI : intuitionistic fuzzy closed H-ideals
- IFnCLHI : intuitionistic fuzzy n-fold closed H-ideals
- FnHI : fuzzy n-fold H-ideals
- NFHI : neutrosophic fuzzy H-ideals

II. Preliminaries

In this section, we give some basic information, which are used in this paper developmentation.

A BCK-A is a non-empty set $\mathfrak{A}$ with a binary operation $\between$ and a constant 0 satisfying the following axioms:

$(\boldsymbol{\mathcal{BCK}1})$ $\big((\mathfrak{f} \between \mathfrak{g}) \between (\mathfrak{f} \between \mathscr{h})\big) \leq (\mathscr{h} \between \mathfrak{g})$,

$(\boldsymbol{\mathcal{BCK}2})$ $\big(\mathfrak{f} \between (\mathfrak{f} \between \mathfrak{g})\big) \leq \mathfrak{g}$,

$(\boldsymbol{\mathcal{BCK}3})$ $\mathfrak{f} \leq \mathfrak{f}$,

$(\boldsymbol{\mathcal{BCK}4})$ $\mathfrak{f} \leq \mathfrak{g}, \mathfrak{g} \leq \mathfrak{f} \Rightarrow \mathfrak{f} = \mathfrak{g}$,

$(\boldsymbol{\mathcal{BCK}5})$ $0 \leq \mathfrak{f}$, where $\mathfrak{f} \leq \mathfrak{g}$ is defined $\mathfrak{f} \between \mathfrak{g} = 0$, for all $\mathfrak{f}, \mathfrak{g}, \mathscr{h} \in \mathfrak{A}$

A non-empty subset $\mathfrak{I}$ of $\mathfrak{A}$ is called an ideal of $\mathfrak{A}$ if $0 \in \mathfrak{I}$ and if $\mathfrak{f} \between \mathfrak{g}, \mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \in \mathfrak{I}$.

We note that if $\mathfrak{f}$ is an ideal $\mathfrak{I}$ of $\mathfrak{A}$ and $\mathfrak{g} \leq \mathfrak{f}$, then $\mathfrak{g} \in \mathfrak{I}$.

An ideal $\mathfrak{I}$ of $\mathfrak{A}$ is called closed, if $0 \between \mathfrak{f} \in \mathfrak{I}$ whenever $\mathfrak{f} \in \mathfrak{I}$.

A mapping $\psi: \mathfrak{A} \to \mathfrak{B}$ BCK-A's is called a homomorphism, if $\psi(\mathfrak{f} \between \mathfrak{g}) = \psi(\mathfrak{f}) \between \psi(\mathfrak{g})$, for all $\mathfrak{f}, \mathfrak{g} \in \mathfrak{A}$.

For any element $\mathfrak{f}, \mathfrak{g} \in \mathfrak{A}$, let us write $\mathfrak{f} \between \mathfrak{g}^k$ for $\big(\ldots\big((\mathfrak{f} \between \mathfrak{g}) \between \mathfrak{g}\big) \between \ldots\big) \between \mathfrak{g}$, where $\mathfrak{g}$ occurs k times.

Definition 2.1. An intuitionistic fuzzy set (IFS) $\mathfrak{M}$ in a non-empty set $\mathfrak{A}$ is an object having the form $\mathfrak{M} = \big\{\big(\mathfrak{f}, \xi_{\mathfrak{M}}(\mathfrak{f}), \varpi_{\mathfrak{M}}(\mathfrak{f})\big) \mid \mathfrak{f} \in \mathfrak{A}\big\}$, where the functions $\xi_{\mathfrak{M}}: \mathfrak{A} \to [0,1]$ and $\varpi_{\mathfrak{M}}: \mathfrak{A} \to [0,1]$ denoted the degree of membership (namely $\xi_{\mathfrak{M}}(\mathfrak{f})$) and the degree of non membership (namely $\varpi_{\mathfrak{M}}(\mathfrak{f})$) of each element $\mathfrak{f} \in \mathfrak{A}$ to the set $\mathfrak{M}$ respectively, and $0 \leq \xi_{\mathfrak{M}}(\mathfrak{f}) + \varpi_{\mathfrak{M}}(\mathfrak{f}) \leq 1$, for all $\mathfrak{f} \in \mathfrak{A}$.

Definition 2.2. An intuitionistic fuzzy set (IFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is called an intuitionistic fuzzy ideal (IFI) of $\mathfrak{A}$, if it satisfies the following axioms:

$(\mathbf{IFI1})$ $\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$,

$(\mathbf{IFI2})$ $\xi_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \xi_{\mathfrak{M}}(\mathfrak{g})\}$,

$(\mathbf{IFI3})$ $\varpi_{\mathfrak{M}}(\mathfrak{f}) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$, for all $\mathfrak{f}, \mathfrak{g} \in \mathfrak{A}$.

Definition 2.3. An intuitionistic fuzzy set (IFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is called an intuitionistic fuzzy closed ideal (IFCLI) of $\mathfrak{A}$, if it satisfies

$(\mathbf{IFI2})$, $(\mathbf{IFI3})$ and the following:

$(\mathbf{IFCLI})$ $\xi_{\mathfrak{M}}(0 \between \mathfrak{f}) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0 \between \mathfrak{f}) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$, for all $\mathfrak{f} \in \mathfrak{A}$.

Definition 2.4. A $(\neq \emptyset)$subset $\mathfrak{I}$ of $\mathfrak{A}$ is called a H-ideal of $\mathfrak{A}$, if $0 \in \mathfrak{I}$ and $\mathfrak{f} \between (\mathfrak{g} \between \mathscr{h}), \mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \between \mathscr{h} \in \mathfrak{I}$.

Definition 2.5. A fuzzy subset ξ of $\mathfrak{A}$ is called a fuzzy H-ideal (FHI) of $\mathfrak{A}$, if $\xi(0) \geq \xi(\mathfrak{f})$ and $\xi(\mathfrak{f} \between \mathscr{h}) \geq \min\big\{\xi\big(\mathfrak{f} \between (\mathfrak{g} \between \mathscr{h})\big), \xi(\mathfrak{g})\big\}$, for all $\mathfrak{f}, \mathfrak{g}, \mathscr{h} \in \mathfrak{A}$.

Definition 2.6. Let ξ be a fuzzy set (FS) of $\mathfrak{A}$. The complement of ξ is denoted by $\bar{\xi}$ and is defined as $\bar{\xi}(\mathfrak{f}) = 1 - \xi(\mathfrak{f})$, for all $\mathfrak{f} \in \mathfrak{A}$.

Definition 2.7. Let $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ be an intuitionistic fuzzy set (IFS) of $\mathfrak{A}$. Then

(i) $\diamond\, \mathfrak{M} = \big(\mathfrak{A}, \xi_{\mathfrak{M}}, \bar{\xi}_{\mathfrak{M}}\big)$ and

(ii) $\Delta\mathfrak{M} = (\mathfrak{A}, \bar{\varpi}_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$

Definition 2.8. An intuitionistic fuzzy set (IFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is called an intuitionistic fuzzy H-ideal (IFHI) of $\mathfrak{A}$, if

$(\mathbf{IFHI1})$ $\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$,

$(\mathbf{IFHI2})$ $\xi_{\mathfrak{M}}(\mathfrak{f} \between \mathscr{h}) \geq \min\big\{\xi_{\mathfrak{M}}\big(\mathfrak{f} \between (\mathfrak{g} \between \mathscr{h})\big), \xi_{\mathfrak{M}}(\mathfrak{g})\big\}$,

$(\mathbf{IFHI3})$ $\varpi_{\mathfrak{M}}(\mathfrak{f} \between \mathscr{h}) \leq \max\big\{\varpi_{\mathfrak{M}}\big(\mathfrak{f} \between (\mathfrak{g} \between \mathscr{h})\big), \varpi_{\mathfrak{M}}(\mathfrak{g})\big\}$, for all $\mathfrak{f}, \mathfrak{g}, \mathscr{h} \in \mathfrak{A}$.

Definition 2.9. An intuitionistic fuzzy set (IFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is called an intuitionistic fuzzy closed H-ideal (IFCLHI) of $\mathfrak{A}$, if it satisfies $(\mathbf{IFHI2})$, $(\mathbf{IFHI3})$ and

$(\mathbf{IFCLHI})$ $\xi_{\mathfrak{M}}(0 \between \mathfrak{f}) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0 \between \mathfrak{f}) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$, for all $\mathfrak{f} \in \mathfrak{A}$.

Definition 2.10. Let $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ be an intuitionistic fuzzy set (IFS) of $\mathfrak{A}$. The set $\mathcal{U}(\xi_{\mathfrak{M}};\, \mathscr{s}) = \{\mathfrak{f} \in \mathfrak{A} \mid \xi_{\mathfrak{M}}(\mathfrak{f}) \geq \mathscr{s}\}$ is called upper $\mathscr{s}$-level of $\xi_{\mathfrak{M}}$ and the set $\mathcal{L}(\varpi_{\mathfrak{M}};\, \mathscr{v}) = \{\mathfrak{f} \in \mathfrak{A} \mid \varpi_{\mathfrak{M}}(\mathfrak{f}) \leq \mathscr{v}\}$ is called lower $\mathscr{v}$-level of $\varpi_{\mathfrak{M}}$.

Definition 2.11. A non-empty sub-set $\mathfrak{I}$ of $\mathfrak{A}$ is called an n-fold H-ideal (nHI) of $\mathfrak{A}$, if

(i) $0 \in \mathfrak{I}$ and

(ii) $\exists$ a fixed $n \in \mathfrak{A}$ such that $\mathfrak{f} \between (\mathfrak{g} \between \hbar^n), \mathfrak{g} \in \mathfrak{I} \Rightarrow \mathfrak{f} \between \hbar^n \in \mathfrak{I}$ for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Note: Every n-fold H-ideal (nHI) is an ideal.

Definition 2.12. A Fuzzy subset ξ of $\mathfrak{A}$ is called a fuzzy n-fold H-ideal (FnHI) of $\mathfrak{A}$, if

(i) $\xi(0) \geq \xi(\mathfrak{f})$ and

(ii) $\exists$ a fixed $n \in \mathfrak{A}$ such that $\xi(\mathfrak{f} \between \hbar^n) \geq \min\{\xi(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \xi(\mathfrak{g})\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Definition 2.13. An intuitionistic fuzzy set (IFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is called an intuitionistic fuzzy n-fold H-ideal (IFnHI) of $\mathfrak{A}$, if

(**IFnHI1**) $\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$, $\exists$ a fixed $n \in \mathfrak{A}$ such that

(**IFnHI2**) $\xi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \xi_{\mathfrak{M}}(\mathfrak{g})\}$,

(**IFnHI3**) $\varpi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Proposition 2.14. Every intuitionistic fuzzy n-fold H-ideal (IFnHI) is an intuitionistic fuzzy ideal (IFI).

Definition 2.15. An intuitionistic fuzzy set (IFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is said to be an intuitionistic fuzzy n-fold closed H-ideal (IFnCLHI) of $\mathfrak{A}$, if it satisfies (**IFnHI2**), (**IFnHI3**) and

(**IFnCLHI**) $\xi_{\mathfrak{M}}(0 \between \mathfrak{f}) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0 \between \mathfrak{f}) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$, for all $\mathfrak{f} \in \mathfrak{A}$.

A Neutrosophic fuzzy set (NFS) $\mathfrak{M}$ in a non-empty set $\mathfrak{A}$ is an object having the form $\mathfrak{M} = \{(\mathfrak{f}, \xi_{\mathfrak{M}}(\mathfrak{f}), \zeta_{\mathfrak{M}}(\mathfrak{f}), \varpi_{\mathfrak{M}}(\mathfrak{f})) \mid \mathfrak{f} \in \mathfrak{A}\}$, where the functions $\xi_{\mathfrak{M}}: \mathfrak{A} \to [0,1]$, $\zeta_{\mathfrak{M}}: \mathfrak{A} \to [0,1]$ and $\varpi_{\mathfrak{M}}: \mathfrak{A} \to [0,1]$ denoted the degree of membership (namely $\xi_{\mathfrak{M}}(\mathfrak{f})$), the degree of indeterminacy membership (namely $\zeta_{\mathfrak{M}}(\mathfrak{f})$) and the degree of non membership (namely $\varpi_{\mathfrak{M}}(\mathfrak{f})$) of each element $\mathfrak{f} \in \mathfrak{A}$ to the set $\mathfrak{M}$ respectively, and $0 \leq \xi_{\mathfrak{M}}(\mathfrak{f}) + \zeta_{\mathfrak{M}}(\mathfrak{f}) + \varpi_{\mathfrak{M}}(\mathfrak{f}) \leq 3$, for all $\mathfrak{f} \in \mathfrak{A}$.

Definition 2.16. A neutrosophic fuzzy set (NFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is called a neutrosophic fuzzy H-ideal (NFHI) of $\mathfrak{A}$, if

(**NFHI1**) $\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$, $\zeta_{\mathfrak{M}}(0) \geq \zeta_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$

(**NFHI2**) $\xi_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar)), \xi_{\mathfrak{M}}(\mathfrak{g})\}$,

(**NFHI3**) $\zeta_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar)), \zeta_{\mathfrak{M}}(\mathfrak{g})\}$,

(**NFHI4**) $\varpi_{\mathfrak{M}}(\mathfrak{f} \between \hbar) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar)), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

III. Neutrosophic Fuzzy n-fold H-ideals of BCK-Algebra

In this section we developed the concept of neutrosophic fuzzy n-fold H-ideals in $\mathfrak{A}$ and also proved their related definition, counter examples and some theorems.

Definition 3.1. A neutrosophic fuzzy set (NFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is called a neutrosophic fuzzy n-fold H-ideal (NFnHI) of $\mathfrak{A}$, if

(**NFnHI1**) $\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$, $\zeta_{\mathfrak{M}}(0) \geq \zeta_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$, $\exists$ a fixed $n \in \mathfrak{A}$ such that

(**NFnHI2**) $\xi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \xi_{\mathfrak{M}}(\mathfrak{g})\}$,

(**NFnHI3**) $\zeta_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \zeta_{\mathfrak{M}}(\mathfrak{g})\}$,

(**NFnHI4**) $\varpi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.

Definition 3.2. A neutrosophic fuzzy set (NFS) $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$ is said to be a neutrosophic fuzzy n-fold closed H-ideal (NFnCLHI) of $\mathfrak{A}$, if it satisfies (**NFnHI2**), (**NFnHI3**), (**NFnHI4**) and

(**NFnCLHI**) $\xi_{\mathfrak{M}}(0 \between \mathfrak{f}) \geq \xi_{\mathfrak{M}}(\mathfrak{f})$, $\zeta_{\mathfrak{M}}(0 \between \mathfrak{f}) \geq \zeta_{\mathfrak{M}}(\mathfrak{f})$ and $\varpi_{\mathfrak{M}}(0 \between \mathfrak{f}) \leq \varpi_{\mathfrak{M}}(\mathfrak{f})$, for all $\mathfrak{f} \in \mathfrak{A}$.

Proposition 3.3. Every NFnHI is a NFI.

Proof: Assume the NFnHI defined as $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ of $\mathfrak{A}$.

Sequently, we obtain $\xi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \xi_{\mathfrak{M}}(\mathfrak{g})\}$,
$\zeta_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \zeta_{\mathfrak{M}}(\mathfrak{g})\}$ and
$\varpi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.
Given any $\mathfrak{f} \in \mathfrak{A}$, $\mathfrak{f} \between 0^n = \mathfrak{f}$, thus with the setting of $\hbar = 0$, we attain
$\xi_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between 0^n)), \xi_{\mathfrak{M}}(\mathfrak{g})\}$
$= \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \xi_{\mathfrak{M}}(\mathfrak{g})\}$,
$\zeta_{\mathfrak{M}}(\mathfrak{f}) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between 0^n)), \zeta_{\mathfrak{M}}(\mathfrak{g})\}$
$= \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \zeta_{\mathfrak{M}}(\mathfrak{g})\}$ and
$\varpi_{\mathfrak{M}}(\mathfrak{f}) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between 0^n)), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$
$= \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{g}), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$.
Accordingly, $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFI of $\mathfrak{A}$.
Note: Suppose $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}$.
Then it is straightforward to see that $\xi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \xi_{\mathfrak{M}}(\mathfrak{f} \between (0 \between \hbar^n))$,
$\zeta_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \zeta_{\mathfrak{M}}(\mathfrak{f} \between (0 \between \hbar^n))$ and
$\varpi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq \varpi_{\mathfrak{M}}(\mathfrak{f} \between (0 \between \hbar^n))$, for all $\mathfrak{f}, \hbar \in \mathfrak{A}$.
Lemma 3.4. If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFHI of $\mathfrak{A}$, then we have the following $\mathfrak{f} \leq \mathfrak{t} \Rightarrow$ $\xi_{\mathfrak{M}}(\mathfrak{f}) \geq \xi_{\mathfrak{M}}(\mathfrak{t})$, $\zeta_{\mathfrak{M}}(\mathfrak{f}) \geq \zeta_{\mathfrak{M}}(\mathfrak{t})$ and $\varpi_{\mathfrak{M}}(\mathfrak{f}) \leq \varpi_{\mathfrak{M}}(\mathfrak{t})$, for all $\mathfrak{f}, \mathfrak{t} \in \mathfrak{A}$.
Proof: Let $\mathfrak{f}, \mathfrak{t} \in \mathfrak{A}$ such that $\mathfrak{f} \leq \mathfrak{t} \Rightarrow \mathfrak{f} \between \mathfrak{t} = 0$.
Consider $\xi_{\mathfrak{M}}(\mathfrak{f}) = \xi_{\mathfrak{M}}(\mathfrak{f} \between 0) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{t} \between 0)), \xi_{\mathfrak{M}}(\mathfrak{t})\}$
$= \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{t}), \xi_{\mathfrak{M}}(\mathfrak{t})\}$
$= \xi_{\mathfrak{M}}(\mathfrak{t})$,
$\zeta_{\mathfrak{M}}(\mathfrak{f}) = \zeta_{\mathfrak{M}}(\mathfrak{f} \between 0) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{t} \between 0)), \zeta_{\mathfrak{M}}(\mathfrak{t})\}$
$= \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{t}), \zeta_{\mathfrak{M}}(\mathfrak{t})\}$
$= \zeta_{\mathfrak{M}}(\mathfrak{t})$, and
$\varpi_{\mathfrak{M}}(\mathfrak{f}) = \varpi_{\mathfrak{M}}(\mathfrak{f} \between 0) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{t} \between 0)), \varpi_{\mathfrak{M}}(\mathfrak{t})\}$
$= \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between \mathfrak{t}), \varpi_{\mathfrak{M}}(\mathfrak{t})\}$
$= \varpi_{\mathfrak{M}}(\mathfrak{t})$.
Theorem 3.5. Let $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ be a NFnHI of $\mathfrak{A}$. Then so is $\Diamond\, \mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \bar{\xi}_{\mathfrak{M}})$.
Proof: We have
$\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathfrak{f}) \Rightarrow 1 - \xi_{\mathfrak{M}}(0) \geq 1 - \xi_{\mathfrak{M}}(\mathfrak{f}) \Rightarrow \bar{\xi}_{\mathfrak{M}}(0) \leq \bar{\xi}_{\mathfrak{M}}(\mathfrak{f})$, for any $\mathfrak{f} \in \mathfrak{A}$.
Consider, for any $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$,
$\xi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \xi_{\mathfrak{M}}(\mathfrak{g})\}$
$\Rightarrow 1 - \xi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{1 - \xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), 1 - \xi_{\mathfrak{M}}\mathfrak{g}\}$
$\Rightarrow \bar{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq 1 - \min\{1 - \xi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), 1 - \xi_{\mathfrak{M}}(\mathfrak{g})\}$
$\Rightarrow \bar{\xi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq \max\{\bar{\xi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \bar{\xi}_{\mathfrak{M}}(\mathfrak{g})\}$.
Similarly, $\zeta_{\mathfrak{M}}(0) \geq \zeta_{\mathfrak{M}}(\mathfrak{f})$ and $\zeta_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \zeta_{\mathfrak{M}}(\mathfrak{g})\}$.
Hence, $\Diamond\, \mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \bar{\xi}_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}$.
Theorem 3.6. Let $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ be a NFnHI of $\mathfrak{A}$. Then so is $\Delta\mathfrak{M} = (\mathfrak{A}, \bar{\varpi}_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$.
Proof: We have
$\varpi_{\mathfrak{M}}(0) \leq \varpi_{\mathfrak{M}}(\mathfrak{f}) \Rightarrow 1 - \varpi_{\mathfrak{M}}(0) \leq 1 - \varpi_{\mathfrak{M}}(\mathfrak{f}) \Rightarrow \bar{\varpi}_{\mathfrak{M}}(0) \geq \bar{\varpi}_{\mathfrak{M}}(\mathfrak{f})$, for any $\mathfrak{f} \in \mathfrak{A}$.
Consider, for any $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$,
$\varpi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \varpi_{\mathfrak{M}}(\mathfrak{g})\}$
$\Rightarrow 1 - \varpi_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \leq \max\{1 - \varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), 1 - \varpi_{\mathfrak{M}}(\mathfrak{g})\}$
$\Rightarrow \bar{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq 1 - \max\{1 - \varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), 1 - \varpi_{\mathfrak{M}}(\mathfrak{g})\}$
$\Rightarrow \bar{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\bar{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \bar{\varpi}_{\mathfrak{M}}(\mathfrak{g})\}$.

Similarly, $\zeta_{\mathfrak{M}}(0) \geq \zeta_{\mathfrak{M}}(\mathfrak{f})$ and $\zeta_{\mathfrak{M}}(\mathfrak{f} \between \hbar^n) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \zeta_{\mathfrak{M}}(\mathfrak{g})\}$.
Hence $\Delta\mathfrak{M} = (\mathfrak{A}, \bar{\varpi}_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}$.
Theorem 3.7. Let $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ be a NFnHI of $\mathfrak{A} \Leftrightarrow \lozenge\, \mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \bar{\xi}_{\mathfrak{M}})$, and $\Delta\mathfrak{M} = (\mathfrak{A}, \bar{\varpi}_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ are NFnHI's of $\mathfrak{A}$.

Proof: Theorem proof is similar to the proofs of Theorems 3.5 & 3.6.
Theorem 3.8. If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnCLHI of $\mathfrak{A}$, then so is $\lozenge\, \mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \bar{\xi}_{\mathfrak{M}})$.
Proof: For any $\mathfrak{f} \in \mathfrak{A}$, we have
$\xi_{\mathfrak{M}}(\mathfrak{f} \between 0) \geq \xi_{\mathfrak{M}}(\mathfrak{f}) \Rightarrow 1 - \xi_{\mathfrak{M}}(\mathfrak{f} \between 0) \geq 1 - \xi_{\mathfrak{M}}(\mathfrak{f})$
$\Rightarrow \bar{\xi}_{\mathfrak{M}}(\mathfrak{f} \between 0) \leq \bar{\xi}_{\mathfrak{M}}(\mathfrak{f})$.
Similarly, $\zeta_{\mathfrak{M}}(\mathfrak{f} \between 0) \geq \zeta_{\mathfrak{M}}(\mathfrak{f})$.
Hence $\lozenge\, \mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \bar{\xi}_{\mathfrak{M}})$ is a NFnCLHI of $\mathfrak{A}$.
Theorem 3.9. If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnCLHI of $\mathfrak{A}$, then so is $\Delta\mathfrak{M} = (\mathfrak{A}, \bar{\varpi}_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$.
Proof: For any $\mathfrak{f} \in \mathfrak{A}$, we have
$\varpi_{\mathfrak{M}}(\mathfrak{f} \between 0) \leq \varpi_{\mathfrak{M}}(\mathfrak{f}) \Rightarrow 1 - \varpi_{\mathfrak{M}}(\mathfrak{f} \between 0) \leq 1 - \varpi_{\mathfrak{M}}(\mathfrak{f})$
$\Rightarrow \bar{\varpi}_{\mathfrak{M}}(\mathfrak{f} \between 0) \geq \bar{\varpi}_{\mathfrak{M}}(\mathfrak{f})$.
Similarly, $\zeta_{\mathfrak{M}}(\mathfrak{f} \between 0) \geq \zeta_{\mathfrak{M}}(\mathfrak{f})$.
Hence $\Delta\mathfrak{M} = (\mathfrak{A}, \bar{\varpi}_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnCLHI of $\mathfrak{A}$.
Theorem 3.10. If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnCLHI of $\mathfrak{A} \Leftrightarrow \diamond\, \mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \bar{\xi}_{\mathfrak{M}})$ and $\Delta\mathfrak{M} = (\mathfrak{A}, \bar{\varpi}_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ are NFnCLHI's of $\mathfrak{A}$.
Theorem 3.11. $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A} \Leftrightarrow$ the non-empty upper s-level cut $\mathcal{U}(\xi_{\mathfrak{M}};\, s)$, t-level cut $\mathcal{U}(\zeta_{\mathfrak{M}};\, t)$ and the non-empty lower v-level cut $\mathcal{L}(\varpi_{\mathfrak{M}};\, v)$ are nHI's of $\mathfrak{A}$, for any $s, t, v \in [0,1]$.
Proof: Suppose $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}$. For any $s, t, v \in [0,1]$, define the sets $\mathcal{U}(\xi_{\mathfrak{M}};\, s) = \{\mathfrak{f} \in \mathfrak{A} \mid \xi_{\mathfrak{M}}(\mathfrak{f}) \geq s\}$, $\mathcal{U}(\zeta_{\mathfrak{M}};\, t) = \{\mathfrak{f} \in \mathfrak{A} \mid \zeta_{\mathfrak{M}}(\mathfrak{f}) \geq t\}$ and $\mathcal{L}(\varpi_{\mathfrak{M}};\, v) = \{\mathfrak{f} \in \mathfrak{A} \mid \varpi_{\mathfrak{M}}(\mathfrak{f}) \leq v\}$. Since $\mathcal{L}(\varpi_{\mathfrak{M}};\, v) \neq \emptyset$, for $\mathfrak{f} \in \mathcal{L}(\varpi_{\mathfrak{M}};\, v)$
$\Rightarrow \varpi_{\mathfrak{M}}(\mathfrak{f}) \leq v \Rightarrow \varpi_{\mathfrak{M}}(0) \leq v \Rightarrow 0 \in \mathcal{L}(\varpi_{\mathfrak{M}};\, v)$. Let $\mathfrak{f} \between (\mathfrak{g} \between \hbar^n) \in \mathcal{L}(\varpi_{\mathfrak{M}};\, v)$ and $\mathfrak{g} \in \mathcal{L}(\varpi_{\mathfrak{M}};\, v) \Rightarrow \varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)) \leq v$ and $\varpi_{\mathfrak{M}}(\mathfrak{g}) \leq v$.
Since, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$, $\varpi_{\mathfrak{M}}(\mathfrak{g} \between \hbar^n) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{f} \between (\mathfrak{g} \between \hbar^n)), \varpi_{\mathfrak{M}}(\mathfrak{g})\} \leq \max\{v, v\} = v$
$\Rightarrow \varpi_{\mathfrak{M}}(\mathfrak{g} \between \hbar^n) \leq v$. Therefore $\mathfrak{g} \between \hbar^n \in \mathcal{L}(\varpi_{\mathfrak{M}};\, v)$, for all $\mathfrak{f}, \mathfrak{g}, \hbar \in \mathfrak{A}$.
Hence $\mathcal{L}(\varpi_{\mathfrak{M}};\, v)$ is nHI of $\mathfrak{A}$. Similarly, we can prove $\mathcal{U}(\xi_{\mathfrak{M}};\, s)$ and $\mathcal{U}(\zeta_{\mathfrak{M}};\, t)$ are nHI's of $\mathfrak{A}$.
Conversely, suppose that $\mathcal{U}(\xi_{\mathfrak{M}};\, s)$, $\mathcal{U}(\zeta_{\mathfrak{M}};\, t)$ and $\mathcal{L}(\varpi_{\mathfrak{M}};\, v)$ are nHI's of $\mathfrak{A}$, for any $s, t, v \in [0,1]$. If possible, assume $\mathfrak{k}, r, c \in \mathfrak{A}$ such that $\xi_{\mathfrak{M}}(0) < \xi_{\mathfrak{M}}(\mathfrak{k})$, $\zeta_{\mathfrak{M}}(0) < \zeta_{\mathfrak{M}}(r)$ and $\varpi_{\mathfrak{M}}(0) > \varpi_{\mathfrak{M}}(c)$. Put, $s_0 = \frac{1}{2}[\xi_{\mathfrak{M}}(0) + \xi_{\mathfrak{M}}(\mathfrak{k})]$
$\Rightarrow s_0 < \xi_{\mathfrak{M}}(\mathfrak{k}),\ 0 \leq \xi_{\mathfrak{M}}(0) < s_0 < 1$
$\Rightarrow \mathfrak{k} \in \mathcal{U}(\xi_{\mathfrak{M}};\, s_0)$.
Since $\mathcal{U}(\xi_{\mathfrak{M}};\, s_0)$ is nHI of $\mathfrak{A}$, we have $0 \in \mathcal{U}(\xi_{\mathfrak{M}};\, s_0) \Rightarrow \xi_{\mathfrak{M}}(0) \geq s_0$, which is a contradiction. Therefore, $\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathfrak{k})$, for all $\mathfrak{k} \in \mathfrak{A}$.
Similarly by taking $t_0 = \frac{1}{2}[\zeta_{\mathfrak{M}}(0) + \zeta_{\mathfrak{M}}(r)]$ and $v_0 = \frac{1}{2}[\varpi_{\mathfrak{M}}(0) + \varpi_{\mathfrak{M}}(c)]$, we can show $\zeta_{\mathfrak{M}}(0) \geq \zeta_{\mathfrak{M}}(r)$ and $\varpi_{\mathfrak{M}}(0) \leq \varpi_{\mathfrak{M}}(c)$, for all $r, c \in \mathfrak{A}$.
If possible assume $\mathfrak{k}, r, c \in \mathfrak{A}$ such that $\xi_{\mathfrak{M}}(\mathfrak{k} \between c^n) < \min\{\xi_{\mathfrak{M}}(\mathfrak{k} \between (r \between c^n)), \xi_{\mathfrak{M}}(r)\}$.
Put, $s_0 = \frac{1}{2}\left[\xi_{\mathfrak{M}}(\mathfrak{k} \between c^n) + \min\{\xi_{\mathfrak{M}}(\mathfrak{k} \between (r \between c^n)), \xi_{\mathfrak{M}}(r)\}\right]$
$\Rightarrow s_0 > \xi_{\mathfrak{M}}(\mathfrak{k} \between c^n)$ and $s_0 < \min\{\xi_{\mathfrak{M}}(\mathfrak{k} \between (r \between c^n)), \xi_{\mathfrak{M}}(r)\}$
$\Rightarrow s_0 > \xi_{\mathfrak{M}}(\mathfrak{k} \between c^n)$, $s_0 < \xi_{\mathfrak{M}}(\mathfrak{k} \between (r \between c^n))$ and $s_0 < \xi_{\mathfrak{M}}(r)$
$\Rightarrow \mathfrak{k} \between c^n \notin \mathcal{U}(\xi_{\mathfrak{M}};\, s_0)$, $\Rightarrow \mathfrak{k} \between (r \between c^n) \in \mathcal{U}(\xi_{\mathfrak{M}};\, s_0)$ and $r \in \mathcal{U}(\xi_{\mathfrak{M}};\, s_0)$,
which is a contradiction to nHI $\mathcal{U}(\xi_{\mathfrak{M}};\, s_0)$.

Therefore, $\xi_{\mathfrak{M}}(\mathfrak{t} \between c^n) \geq \min\{\xi_{\mathfrak{M}}(\mathfrak{t} \between (\mathscr{r} \between c^n)), \xi_{\mathfrak{M}}(\mathscr{r})\}$, for any $\mathfrak{t}, \mathscr{r}, c \in \mathfrak{A}$. Similarly we can prove $\zeta_{\mathfrak{M}}(\mathfrak{t} \between c^n) \geq \min\{\zeta_{\mathfrak{M}}(\mathfrak{t} \between (\mathscr{r} \between c^n)), \zeta_{\mathfrak{M}}(\mathscr{r})\}$ and $\varpi_{\mathfrak{M}}(\mathfrak{t} \between c^n) \leq \max\{\varpi_{\mathfrak{M}}(\mathfrak{t} \between (\mathscr{r} \between c^n)), \varpi_{\mathfrak{M}}(\mathscr{r})\}$, for any $\mathfrak{t}, \mathscr{r}, c \in \mathfrak{A}$. Hence $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}$.

Theorem 3.12. If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnCLHI of $\mathfrak{A} \Leftrightarrow$ the non-empty upper $\mathscr{s}$-level cut $\mathcal{U}(\xi_{\mathfrak{M}}; \mathscr{s})$, upper $\mathscr{t}$-level cut $\mathcal{U}(\zeta_{\mathfrak{M}}; \mathscr{t})$ and the non-empty lower $\mathscr{v}$-level cut $\mathcal{L}(\varpi_{\mathfrak{M}}; \mathscr{v})$ are nCLHI's of $\mathfrak{A}$, for any $\mathscr{s}, \mathscr{t}, \mathscr{v} \in [0,1]$.

Corollary 3.13. If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}$, then the sets
$\mathcal{A} = \{\mathbb{f} \in \mathfrak{A} | \ \xi_{\mathfrak{M}}(\mathbb{f}) = \xi_{\mathfrak{M}}(0)\}$, $\mathbb{M} = \{\mathbb{f} \in \mathfrak{A} | \ \zeta_{\mathfrak{M}}(\mathbb{f}) = \zeta_{\mathfrak{M}}(0)\}$ and
$\mathbb{R} = \{\mathbb{f} \in \mathfrak{A} | \ \varpi_{\mathfrak{M}}(\mathbb{f}) = \varpi_{\mathfrak{M}}(0)\}$ are nHI's of $\mathfrak{A}$.

Proof: Since $0 \in \mathfrak{A}$, $\xi_{\mathfrak{M}}(0) = \xi_{\mathfrak{M}}(0)$, $\zeta_{\mathfrak{M}}(0) = \zeta_{\mathfrak{M}}(0)$ and $\varpi_{\mathfrak{M}}(0) = \varpi_{\mathfrak{M}}(0)$
$\Rightarrow$ $0 \in \mathcal{A}$, $0 \in \mathbb{M}$ and $0 \in \mathbb{R}$, so $\mathcal{A} \neq \emptyset$, $\mathbb{M} \neq \emptyset$ and $\mathbb{R} \neq \emptyset$.
Let $\mathbb{f} \between (\mathbb{g} \between \mathscr{h}^n) \in \mathcal{A}$ and $\mathbb{g} \in \mathcal{A}$
$\Rightarrow \xi_{\mathfrak{M}}(\mathbb{f} \between (\mathbb{g} \between \mathscr{h}^n)) = \xi_{\mathfrak{M}}(0)$ and $\xi_{\mathfrak{M}}(\mathbb{g}) = \xi_{\mathfrak{M}}(0)$.
Since $\xi_{\mathfrak{M}}(\mathbb{f} \between \mathscr{h}^n) \geq \min\{\xi_{\mathfrak{M}}(\mathbb{f} \between (\mathbb{g} \between \mathscr{h}^n)), \xi_{\mathfrak{M}}(\mathbb{g})\} = \xi_{\mathfrak{M}}(0)$
$\Rightarrow \xi_{\mathfrak{M}}(\mathbb{f} \between \mathscr{h}^n) \geq \xi_{\mathfrak{M}}(0)$, but $\xi_{\mathfrak{M}}(0) \geq \xi_{\mathfrak{M}}(\mathbb{f} \between \mathscr{h}^n)$.
It follows that $\mathbb{f} \between \mathscr{h}^n \in \mathcal{A}$, for all $\mathbb{f}, \mathbb{g}, \mathscr{h} \in \mathfrak{A}$.
Hence $\mathcal{A}$ is nHI of $\mathfrak{A}$. Similarly we can prove $\mathbb{M}$, $\mathbb{R}$ are nHI of $\mathfrak{A}$.

Definition 3.14. Let $\psi: \mathfrak{A} \to \mathfrak{A}'$ be a homomorphism of $\mathfrak{A}$. For any NFS $\mathfrak{M} = (\mathfrak{A}', \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$, we define a new NFS $\mathfrak{M}^{\psi} = (\mathfrak{A}, \xi_{\mathfrak{M}}^{\psi}, \zeta_{\mathfrak{M}}^{\psi}, \varpi_{\mathfrak{M}}^{\psi})$ in $\mathfrak{A}$ by $\xi_{\mathfrak{M}}^{\psi}(\mathbb{f}) = \xi_{\mathfrak{M}}(\psi(\mathbb{f}))$, $\zeta_{\mathfrak{M}}^{\psi}(\mathbb{f}) = \zeta_{\mathfrak{M}}(\psi(\mathbb{f}))$ and $\varpi_{\mathfrak{M}}^{\psi}(\mathbb{f}) = \varpi_{\mathfrak{M}}(\psi(\mathbb{f}))$, for all $\mathbb{f} \in \mathfrak{A}$.

Theorem 3.15. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be BCK-A's and ψ a homomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.
(i) If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}'$ then $\mathfrak{M}^{\psi} = (\mathfrak{A}, \xi_{\mathfrak{M}}^{\psi}, \zeta_{\mathfrak{M}}^{\psi}, \varpi_{\mathfrak{M}}^{\psi})$ is a NFnHI of $\mathfrak{A}$.
(ii) If $\mathfrak{M}^{\psi} = (\mathfrak{A}, \xi_{\mathfrak{M}}^{\psi}, \zeta_{\mathfrak{M}}^{\psi}, \varpi_{\mathfrak{M}}^{\psi})$ is a NFnHI of $\mathfrak{A}$ then $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}'$.

Proof: (i) For $\mathbb{f}' \in \mathfrak{A}'$ $\exists$ $\mathbb{f} \in \mathfrak{A}$ such that $\mathscr{f}(\mathbb{f}) = \mathbb{f}'$, we have
$\xi_{\mathfrak{M}}^{\psi}(0) = \xi_{\mathfrak{M}}(\psi(0)) = \xi_{\mathfrak{M}}(0') \geq \xi_{\mathfrak{M}}(\mathbb{f}') = \xi_{\mathfrak{M}}(\psi(\mathbb{f})) = \xi_{\mathfrak{M}}^{\psi}(\mathbb{f})$,
$\zeta_{\mathfrak{M}}^{\psi}(0) = \zeta_{\mathfrak{M}}(\psi(0)) = \zeta_{\mathfrak{M}}(0') \geq \zeta_{\mathfrak{M}}(\mathbb{f}') = \zeta_{\mathfrak{M}}(\psi(\mathbb{f})) = \zeta_{\mathfrak{M}}^{\psi}(\mathbb{f})$, and
$\varpi_{\mathfrak{M}}^{\psi}(0) = \varpi_{\mathfrak{M}}(\psi(0)) = \varpi_{\mathfrak{M}}(0') \leq \varpi_{\mathfrak{M}}(\mathbb{f}') = \varpi_{\mathfrak{M}}(\psi(\mathbb{f})) = \varpi_{\mathfrak{M}}^{\psi}(\mathbb{f})$.
Let $\mathbb{f}, \mathbb{g} \in \mathfrak{A}$, $\mathbb{g}' \in \mathfrak{A}'$ then $\exists$ $\mathbb{g} \in \mathfrak{A}$ such that $\psi(\mathbb{g}) = \mathbb{g}'$. We have

$$\begin{aligned}
\xi_{\mathfrak{M}}^{\psi}(\mathbb{f} \between \mathscr{h}^n) &= \xi_{\mathfrak{M}}(\psi(\mathbb{f} \between \mathscr{h}^n)) = \xi_{\mathfrak{M}}(\psi(\mathbb{f}) \between (\psi(\mathscr{h}))^n) \\
&\geq \min\left\{\xi_{\mathfrak{M}}\left(\psi(\mathbb{f}) \between \left(\mathbb{g}' \between (\psi(\mathscr{h}))^n\right)\right), \xi_{\mathfrak{M}}(\mathbb{g}')\right\}, \\
&= \min\left\{\xi_{\mathfrak{M}}\left(\psi(\mathbb{f}) \between \left(\psi(\mathbb{g}) \between (\psi(\mathscr{h}))^n\right)\right), \xi_{\mathfrak{M}}(\psi(\mathbb{g}))\right\}, \\
&= \min\{\xi_{\mathfrak{M}}^{\psi}(\mathbb{f} \between (\mathbb{g} \between \mathscr{h}^n)), \xi_{\mathfrak{M}}^{\psi}(\mathbb{g})\},
\end{aligned}$$

$$\begin{aligned}
\zeta_{\mathfrak{M}}^{\psi}(\mathbb{f} \between \mathscr{h}^n) &= \zeta_{\mathfrak{M}}(\psi(\mathbb{f} \between \mathscr{h}^n)) = \zeta_{\mathfrak{M}}(\psi(\mathbb{f}) \between (\psi(\mathscr{h}))^n) \\
&\geq \min\left\{\zeta_{\mathfrak{M}}\left(\psi(\mathbb{f}) \between \left(\mathbb{g}' \between (\psi(\mathscr{h}))^n\right)\right), \zeta_{\mathfrak{M}}(\mathbb{g}')\right\}, \\
&= \min\left\{\zeta_{\mathfrak{M}}\left(\psi(\mathbb{f}) \between \left(\psi(\mathbb{g}) \between (\psi(\mathscr{h}))^n\right)\right), \zeta_{\mathfrak{M}}(\psi(\mathbb{g}))\right\}, \\
&= \min\{\zeta_{\mathfrak{M}}^{\psi}(\mathbb{f} \between (\mathbb{g} \between \mathscr{h}^n)), \zeta_{\mathfrak{M}}^{\psi}(\mathbb{g})\},
\end{aligned}$$

and

$$\begin{aligned}
\varpi_{\mathfrak{M}}^{\psi}(\mathbb{f} \between \mathscr{h}^n) &= \varpi_{\mathfrak{M}}(\psi(\mathbb{f} \between \mathscr{h}^n)) = \varpi_{\mathfrak{M}}(\psi(\mathbb{f}) \between (\psi(\mathscr{h}))^n) \\
&\leq \max\left\{\varpi_{\mathfrak{M}}\left(\psi(\mathbb{f}) \between \left(\mathbb{g}' \between (\psi(\mathscr{h}))^n\right)\right), \varpi_{\mathfrak{M}}(\mathbb{g}')\right\}, \\
&= \max\left\{\varpi_{\mathfrak{M}}\left(\psi(\mathbb{f}) \between \left(\psi(\mathbb{g}) \between (\psi(\mathscr{h}))^n\right)\right), \varpi_{\mathfrak{M}}(\psi(\mathbb{g}))\right\}, \\
&= \max\{\varpi_{\mathfrak{M}}^{\psi}(\mathbb{f} \between (\mathbb{g} \between \mathscr{h}^n)), \varpi_{\mathfrak{M}}^{\psi}(\mathbb{g})\}.
\end{aligned}$$

Hence $\mathfrak{M}^{\psi} = \left(\mathfrak{A}, \xi_{\mathfrak{M}}^{\psi}, \zeta_{\mathfrak{M}}^{\psi}, \varpi_{\mathfrak{M}}^{\psi}\right)$ is a NFnHI of $\mathfrak{A}$.

(ii) Since $\psi: \mathfrak{A} \to \mathfrak{A}'$ onto, for $\mathbb{f}, \mathbb{g}, \hbar \in \mathfrak{A}'$ $\exists\ \mathfrak{k}, r, c \in \mathfrak{A}$ such that $\psi(\mathfrak{k}) = d$, $\psi(r) = \mathfrak{e}$ and $\psi(c) = \mathbb{f}$.

$$\begin{aligned}
\xi_{\mathfrak{M}}(\mathbb{f} \between \hbar^n) &= \xi_{\mathfrak{M}}\left(\psi(\mathfrak{k}) \between \left(\psi(c)\right)^n\right) = \xi_{\mathfrak{M}}\left(\psi(\mathfrak{k} \between c^n)\right) = \xi_{\mathfrak{M}}^{\psi}(\mathfrak{k} \between c^n) \\
&\geq \min\left\{\xi_{\mathfrak{M}}^{\psi}\left(\mathfrak{k} \between (r \between c^n)\right), \xi_{\mathfrak{M}}^{\psi}(r)\right\}, \\
&= \min\left\{\xi_{\mathfrak{M}}\left(\psi(\mathfrak{k}) \between \left(\psi(r) \between \left(\psi(c)\right)^n\right)\right), \xi_{\mathfrak{M}}\left(\psi(r)\right)\right\}, \\
&= \min\left\{\xi_{\mathfrak{M}}\left(\mathbb{f} \between (\mathbb{g} \between \hbar^n)\right), \xi_{\mathfrak{M}}(\mathbb{g})\right\},
\end{aligned}$$

$$\begin{aligned}
\zeta_{\mathfrak{M}}(\mathbb{f} \between \hbar^n) &= \zeta_{\mathfrak{M}}\left(\psi(\mathfrak{k}) \between \left(\psi(c)\right)^n\right) = \zeta_{\mathfrak{M}}\left(\psi(\mathfrak{k} \between c^n)\right) = \zeta_{\mathfrak{M}}^{\psi}(\mathfrak{k} \between c^n) \\
&\geq \min\left\{\zeta_{\mathfrak{M}}^{\psi}\left(\mathfrak{k} \between (r \between c^n)\right), \zeta_{\mathfrak{M}}^{\psi}(r)\right\}, \\
&= \min\left\{\zeta_{\mathfrak{M}}\left(\psi(\mathfrak{k}) \between \left(\psi(r) \between \left(\psi(c)\right)^n\right)\right), \zeta_{\mathfrak{M}}\left(\psi(r)\right)\right\}, \\
&= \min\left\{\zeta_{\mathfrak{M}}\left(\mathbb{f} \between (\mathbb{g} \between \hbar^n)\right), \zeta_{\mathfrak{M}}(\mathbb{g})\right\}
\end{aligned}$$

and

$$\begin{aligned}
\varpi_{\mathfrak{M}}(\mathbb{f} \between \hbar^n) &= \varpi_{\mathfrak{M}}\left(\psi(\mathfrak{k}) \between \left(\psi(c)\right)^n\right) = \varpi_{\mathfrak{M}}\left(\psi(\mathfrak{k} \between c^n)\right) = \varpi_{\mathfrak{M}}^{\psi}(\mathfrak{k} \between c^n) \\
&\leq \max\left\{\varpi_{\mathfrak{M}}^{\psi}\left(\mathfrak{k} \between (r \between c^n)\right), \varpi_{\mathfrak{M}}^{\psi}(r)\right\}, \\
&= \max\left\{\varpi_{\mathfrak{M}}\left(\psi(\mathfrak{k}) \between \left(\psi(r) \between \left(\psi(c)\right)^n\right)\right), \varpi_{\mathfrak{M}}\left(\psi(r)\right)\right\}, \\
&= \max\left\{\varpi_{\mathfrak{M}}\left(\mathbb{f} \between (\mathbb{g} \between \hbar^n)\right), \varpi_{\mathfrak{M}}(\mathbb{g})\right\}.
\end{aligned}$$

Hence $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NFnHI of $\mathfrak{A}'$.

Definition 3.16. Let $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ be a neutrosophic fuzzy set (NFS) of $\mathfrak{A}$ then we say that $\xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}$ has 'Sup' property, if for any subset $\mathbb{S} \subseteq \mathfrak{A}$, $\mathbb{T} \subseteq \mathfrak{A}$ there exists $\mathbb{f}_0 \in \mathbb{S}$, $\mathbb{g}_0 \in \mathbb{T}$ such that $\xi_{\mathfrak{M}}(\mathbb{f}_0) = \underset{s \in \mathbb{S}}{\text{Sup}}\, \xi_{\mathfrak{M}}(s)$, $\zeta_{\mathfrak{M}}(\mathbb{g}_0) = \underset{t \in \mathbb{T}}{\text{Sup}}\, \zeta_{\mathfrak{M}}(t)$, and we say that $\varpi_{\mathfrak{M}}$ has ' inf ' property, if for any subset $\mathbb{V} \subseteq \mathfrak{A}$ there exists $\hbar_0 \in \mathbb{V}$ such that $\varpi_{\mathfrak{M}}(\hbar_0) = \underset{v \in \mathbb{V}}{\inf}\, \varpi_{\mathfrak{M}}(v)$.

Theorem 3.17. Let $\psi: \mathfrak{A} \to \mathfrak{A}'$ be an onto homomorphism of $\mathfrak{A}$. If $\mathfrak{M} = (\mathfrak{A}, \xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}, \varpi_{\mathfrak{M}})$ is a NnHI of $\mathfrak{A}$ with $\xi_{\mathfrak{M}}, \zeta_{\mathfrak{M}}$ has 'sup' property and $\varpi_{\mathfrak{M}}$ has ' inf ' property then the image of $\mathfrak{M}$ under ψ is also NnHI of $\mathfrak{A}'$.

IV. Conclusion

We successfully discussed Definitions and counter examples of neutrosophic fuzzy n-fold H-ideal of BCK-algebra, and we give some necessary and sufficient conditions related to H-ideal and closed H-ideals of BCK-algebra and also discussed their related properties.

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